*Proof:*

$$\nabla \cdot (\mathbf{F} \wedge \mathbf{G}) = \sum \left\{ \mathbf{i} \cdot \frac{\partial}{\partial x} (\mathbf{F} \wedge \mathbf{G}) \right\}$$

$$= \sum \left\{ \mathbf{i} \cdot \left[ \mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x} + \frac{\partial \mathbf{F}}{\partial x} \wedge \mathbf{G} \right] \right\}$$

$$= \sum \left[ \mathbf{i}, \mathbf{F}, \frac{\partial \mathbf{G}}{\partial x} \right] + \sum \left[ \mathbf{i}, \frac{\partial \mathbf{F}}{\partial x}, \mathbf{G} \right]$$

$$= - \sum \left[ \mathbf{F}, \mathbf{i}, \frac{\partial \mathbf{G}}{\partial x} \right] + \sum \left[ \left( \mathbf{i} \wedge \frac{\partial \mathbf{F}}{\partial x} \right) \cdot \mathbf{G} \right]$$

$$= - \mathbf{F} \cdot \sum \left( \mathbf{i} \wedge \frac{\partial \mathbf{G}}{\partial x} \right) + \left\{ \sum \left( \mathbf{i} \wedge \frac{\partial \mathbf{F}}{\partial x} \right) \right\} \cdot \mathbf{G}$$

$$= \underline{- \mathbf{F} \cdot \operatorname{curl} \mathbf{G} + \mathbf{G} \cdot \operatorname{curl} \mathbf{F}.}$$

(v) curl $(\mathbf{F} \wedge \mathbf{G}) \equiv (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F}$.

*Proof:*

$$\nabla \wedge (\mathbf{F} \wedge \mathbf{G}) = \sum \left\{ \mathbf{i} \wedge \frac{\partial}{\partial x} (\mathbf{F} \wedge \mathbf{G}) \right\}$$

$$= \sum \left\{ \mathbf{i} \wedge \left( \mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x} + \frac{\partial \mathbf{F}}{\partial x} \wedge \mathbf{G} \right) \right\}$$

$$= \sum \left\{ \mathbf{i} \wedge \left( \mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x} \right) \right\} + \sum \left\{ \mathbf{i} \wedge \left( \frac{\partial \mathbf{F}}{\partial x} \wedge \mathbf{G} \right) \right\}$$

$$= \sum \left\{ \left( \mathbf{i} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) \mathbf{F} - (\mathbf{i} \cdot \mathbf{F}) \frac{\partial \mathbf{G}}{\partial x} \right\} + \sum \left\{ (\mathbf{i} \cdot \mathbf{G}) \frac{\partial \mathbf{F}}{\partial x} - \left( \mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x} \right) \mathbf{G} \right\}$$

$$= \left( \sum \mathbf{i} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) \mathbf{F} - \mathbf{F} \cdot \left( \sum \mathbf{i} \frac{\partial}{\partial x} \right) \mathbf{G} + \mathbf{G} \cdot \left( \sum \mathbf{i} \frac{\partial}{\partial x} \right) \mathbf{F} - \left( \sum \mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x} \right) \mathbf{G}$$

$$= \underline{(\nabla \cdot \mathbf{G})\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G}.}$$

(*N.B.* In this example naïve application of the vector product rule would give the wrong result:

sometimes convenient to work with other systems of orthogonal coordinates. The two most common are those afforded by *cylindrical polar coordinates* and *spherical polar coordinates*.

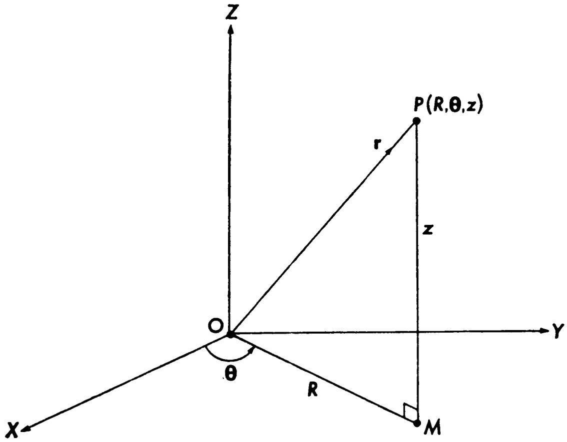


FIG. 1.24

Figure 1.24 shows a point $P$ in a Cartesian tri-rectangular frame specified by axes $OX$, $OY$, $OZ$. $PM$ is drawn perpendicularly to the coordinate plane $XOY$. Let $XOM=\theta$, $OM=R$, $PM=z$. Then $(R, \theta, z)$ are called the cylindrical polar coordinates of $P$. Knowledge of these elements clearly enables $P$ to be determined uniquely. The three surfaces through $P$

$$R=\text{const.},\quad \theta=\text{const.},\quad z=\text{const.}$$

are respectively (i) the right circular cylinder through $P$, axis $OX$ and radius $R$; (ii) the plane through $OZ$ set at angle $\theta$ to the coordinate plane $ZOX$; (iii) the plane through $P$ parallel to the coordinate plane $XOY$ and distant $z$ from it. Clearly these three surfaces intersect mutually orthogonally, so that the system of coordinates is an orthogonal one. We observe that if $(x, y, z)$ are the Cartesian coordinates of $P$, then

$$x=R\cos\theta,\quad y=R\sin\theta,\quad z=z;$$
$$R=\sqrt{(x^2+y^2)},\quad \theta=\tan^{-1}(y/x),\quad z=z.$$

The first system of equations expresses the Cartesian coordinates in terms of the cylindrical polars: the second system gives the cylindrical polar coordinates in terms of the Cartesians.

In Fig. 1.25, $P$ is the point chosen in a Cartesian tri-rectangular frame of axes $OX$, $OY$, $OZ$, such that $OP=r$, $\angle ZOP=\theta$, $\varphi=$angle between planes $ZOX$, $ZOP$. Then $(r, \theta, \varphi)$ are called the spherical polar coordinates of $P$. Knowledge of these elements clearly enables $P$ to be determined uniquely. The three surfaces through $P$

$$r=\text{const.},\quad \theta=\text{const.},\quad \varphi=\text{const.}$$

$\rho$ being a scalar field and $c$ a constant. Assuming that **E**, **H**, **v** and $\rho$ are at least twice differentiable functions of $x$, $y$, $z$ and $t$, establish the following equations:

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \nabla\rho + \frac{1}{c^2}\frac{\partial}{\partial t}(\rho\mathbf{v}),$$

$$\nabla^2\mathbf{H} - \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = -\frac{1}{c}\nabla\wedge(\rho\mathbf{v}),$$

$$\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t} = 0.$$

(London Univ., Gen. B.Sc. II, 1963)

15. (i) Show that

$$\operatorname{div}(\mathbf{u}\wedge\mathbf{v}) = \mathbf{v}\cdot\operatorname{curl}\mathbf{u} - \mathbf{u}\cdot\operatorname{curl}\mathbf{v}$$

for any two vector fields **u** and **v**.

If curl $\mathbf{a} = \mathbf{b}$, and curl $\mathbf{b} = \mathbf{a}$, show that

(a) div (curl $\mathbf{a} \wedge$ curl $\mathbf{b}$) $= \mathbf{a}^2 - \mathbf{b}^2$,

(b) $\nabla^2\mathbf{a} + \mathbf{a} = \mathbf{0}$, $\nabla^2\mathbf{b} + \mathbf{b} = \mathbf{0}$.

(ii) If $\theta$, $\varphi$, $\psi$ are suitable functions of position, prove that

$$\int_V (\psi\nabla\theta\cdot\nabla\varphi)\,dV = \int_S (\varphi\psi\nabla\theta)\cdot d\mathbf{S} - \int_V \left\{\varphi\nabla\cdot(\psi\nabla\theta)\right\}dV$$

$$= \int_S (\theta\psi\nabla\varphi)\cdot d\mathbf{S} - \int_V \left\{\theta\nabla\cdot(\psi\nabla\varphi)\right\}dV,$$

where $S$ is a simple closed surface bounding a volume $V$.

(Battersea College of Technology, Dip. Tech. I, 1964)

16. Using Green's theorem that

$$\iint_S \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)dS = \iiint_\tau (U\nabla^2 V - V\nabla^2 U)\,d\tau,$$

show that, if $P$ is any point of a volume $\tau$ bounded by a surface $S$,

$$4\pi V_P = \iint_S \left[\frac{1}{R}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right]dS - \iiint_\tau \frac{1}{R}\nabla^2 V\,d\tau,$$

where $V$ is a scalar function of position satisfying conditions which should be stated, $V_P$ is the value of $V$ at $P$, and $R$ is the distance from $P$ to a variable point of $c$. Hence (i) show that

$$4\pi = -\iint_S \frac{\partial}{\partial n}\left(\frac{1}{R}\right)dS,$$

so comparing this with $\partial\varphi/\partial y = -k^2x/(x^2+y^2)$, we see that

$$f'(y)=0$$

or

$$f(y)=\text{const.}$$

As the constant is immaterial, we may take

$$\varphi(x, y)=k^2 \tan^{-1}(x/y).$$

The equipotentials are thus given by the planes

$$x=cy,$$

through the $z$-axis. They are appropriately intersected orthogonally by the streamlines. Figure 2.6 shows the streamlines and equipotentials.

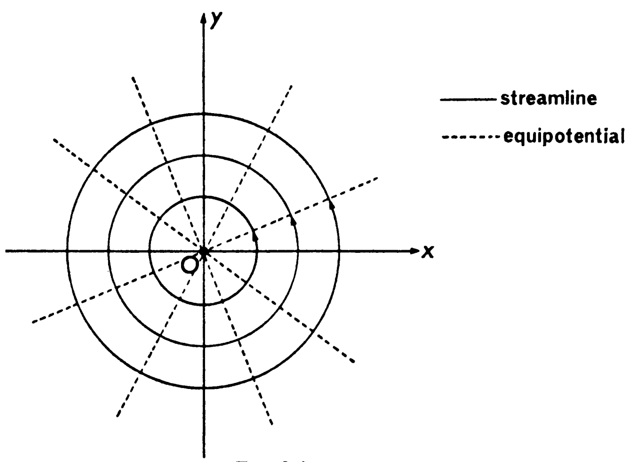


FIG. 2.6

EXAMPLE 2

For an incompressible fluid, $\mathbf{q}=[-\omega y, \omega x, 0]$ ($\omega=$const.). Discuss the nature of the flow.

We find $\nabla \cdot \mathbf{q}=0$, so that such a flow is possible.

Further,

$$\nabla \wedge \mathbf{q}=\begin{Vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{Vmatrix}=2\omega\mathbf{k}.$$

Thus the flow is not of the potential kind. It can easily be shown that a rigid body rotating about the $z$-axis with constant vector angular velocity $\omega\mathbf{k}$ gives the same type of motion. (For the velocity at $(x, y, z)$ in the body is $-\omega y\mathbf{i}+\omega x\mathbf{j}$.)

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