As a special case, let us assume that the y and z axes are principal axes. Then $I_{yz} = 0$ and Eqs. (9-19) and (9-20) reduce to

$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$\tan \phi = \frac{y}{z} = \frac{M_y I_z}{M_z I_y}$$

The first of these equations is the same as Eq. (9-12a), and the second agrees with Eq. (9-14) if we note that $\tan \theta = M_y/M_z$.

Example

An angle section with unequal legs (L $6 \times 4 \times \frac{1}{2}$) is subjected to a bending moment $M_z = 10$ in.-k acting in the xy plane (Fig. 9-15). Determine the maximum tensile and compressive stresses in the beam using the generalized flexure formula.

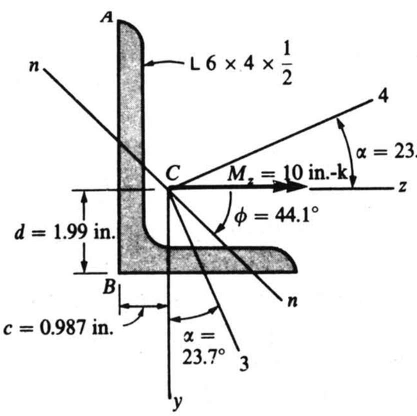


Fig. 9-15 Example. Angle section with unequal legs

The properties of the angle section are given in Appendix E, Table E-5. The centroidal axes y and z are located at distance $c = 0.987$ in. from the back of the long leg and distance $d = 1.99$ in. from the back of the short leg. The corresponding moments of inertia are

$$I_y = 6.27 \text{ in.}^4 \qquad I_z = 17.4 \text{ in.}^4$$

as obtained from the table.

In order to use the generalized flexure formula, we also need the product of inertia I_{yz}. Because it is not a tabulated quantity, we must calculate I_{yz} from the other properties. One method is to use the formula for rotation of axes for products of inertia (Eq. C-15b, Appendix C). We must apply this formula to a rotation from the principal axes to the yz axes, and then, because the product of inertia is zero for the principal axes, we can solve for I_{yz}. The orientation of the principal axes 3-4 is given in Table E-5 by the angle α. For the angle section in this problem, we obtain

$$\alpha = \arctan 0.440 = 23.7°$$

where M_z is the bending moment about the z axis and y is the ordinate to the point under consideration.

Now consider an element $abcd$ cut out between two cross sections, a distance dx apart, and having length s measured along the centerline of the cross section (Fig. 9-21a). The resultant of the normal stresses acting on the face ad is denoted F_1 (Fig. 9-21c) and the resultant on the face bc is denoted F_2. Since the bending moment at face ad is larger than at bc, the force F_1 will be larger than F_2; hence, shear stresses τ must act along the face cd in order to have static equilibrium of the element. These shear stresses must be parallel to the top and bottom surfaces of the element, which are free of stress, and must be accompanied by complementary shear stresses acting on the cross sections ad and bc. Summing forces in the x direction for the element $abcd$ (Fig. 9-21c), we get

$$\tau t\,dx = F_1 - F_2 \tag{a}$$

where t is the thickness of the cross section at cd; that is, t is the thickness at distance s from the free edge of the cross section (Fig. 9-21b). Using Eq. (9-26), we conclude that

$$F_1 = \int_0^s \sigma_x\,dA = -\frac{M_{z1}}{I_z}\int_0^s y\,dA$$

where dA is an element of area on the side ad of the element, y is the coordinate of the element dA, and M_{z1} is the bending moment at this cross section. An analogous expression is obtained for the force F_2:

$$F_2 = \int_0^s \sigma_x\,dA = -\frac{M_{z2}}{I_z}\int_0^s y\,dA$$

Substituting the expressions for F_1 and F_2 into Eq. (a), we get

$$\tau = \frac{M_{z2} - M_{z1}}{dx}\,\frac{1}{I_z t}\int_0^s y\,dA$$

The quantity $(M_{z2} - M_{z1})/dx$ is the rate of change of the bending moment and is equal to $-V_y$, where V_y is the shear force in the y direction (equal to P in Fig. 9-21). Therefore, the equation for the shear stresses is

$$\tau = -\frac{V_y}{I_z t}\int_0^s y\,dA$$

This equation gives the shear stresses at any point in the cross section at distance s from the free edge. The integral on the right-hand side represents the first moment with respect to the neutral axis (the z axis) of the area of the cross section from $s = 0$ to $s = s$. Denoting this first moment by Q_z, and using only the absolute value of the shear stress because its direction can be determined by inspection, we can write the

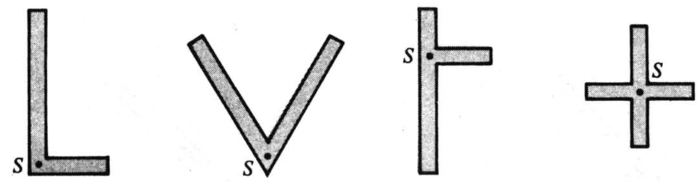


Fig. 9-25 Shear centers S of sections consisting of two intersecting narrow rectangles

For all cross sections consisting of two narrow intersecting rectangles, as in the examples of Fig. 9-25, the shear stresses have resultant forces that intersect at the junction of the rectangles. Therefore, the shear center S is located at the junction, as shown in the figure.

The locations of the shear centers for most structural shapes are given in this section, either in the preceding discussions or in the examples and problems that follow.*

Example 1

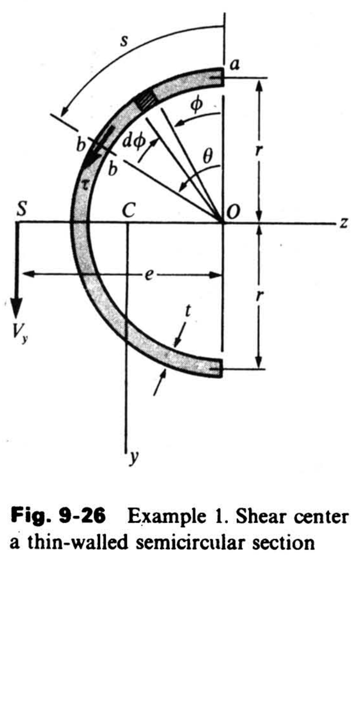


Fig. 9-26 Example 1. Shear center of a thin-walled semicircular section

Locate the shear center S of the thin-walled semicircular cross section shown in Fig. 9-26.

Let us consider a section bb defined by the distance s measured along the median line of the cross section. The central angle subtended between point a, which is at the edge of the section, and section bb is denoted by θ. Therefore, we have $s = r\theta$, where r is the radius of the median line. The first moment of the area between a and section bb is

$$Q_z = \int y\,dA = \int_0^\theta (r\cos\phi)(rt)\,d\phi = r^2 t \sin\theta$$

where t is the thickness of the section. Thus, the shear stress τ at section bb is

$$\tau = \frac{V_y Q_z}{I_z t} = \frac{V_y r^2 \sin\theta}{I_z}$$

Substituting $I_z = \pi r^3 t/2$, we get

$$\tau = \frac{2V_y \sin\theta}{\pi r t} \tag{9-46}$$

When $\theta = 0$ or π, this expression gives $\tau = 0$; and, when $\theta = \pi/2$, it gives the maximum shear stress.

The moment about the center O due to the shear stresses τ is

$$T = \int \tau r\,dA = \int_0^\pi \frac{2V_y r \sin\theta\,d\theta}{\pi} = \frac{4rV_y}{\pi}$$

which must be the same as the moment due to the force V_y acting at the shear center; hence,

$$T = V_y e = \frac{4rV_y}{\pi}$$

* The first determination of a shear center was made by Timoshenko in 1913 (Ref. 9-1). For additional information and for the historical development of the shear-center concept, see Refs. 9-1 through 9-20.

9.3-1 A channel section is subjected to a bending couple M having its vector at an angle θ to the z axis (see figure). Calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Use the following data: C 8×11.5 section, $M = 30$ in.-k, $\tan\theta = \frac{1}{3}$.

9.3-2 Solve the preceding problem for a C 6×13 channel section with $M = 5.0$ in.-k and $\theta = 15°$.

9.3-3 An angle section with equal legs is subjected to a bending moment M having its vector along the 1-1 axis, as shown in the figure. Calculate the maximum tensile stress σ_t and maximum compressive stress σ_c if the angle is an L $6 \times 6 \times \frac{3}{4}$ section and $M = 20$ in.-k.

9.3-4 Solve the preceding problem for an L $4 \times 4 \times \frac{1}{2}$ angle section with $M = 6$ in.-k.

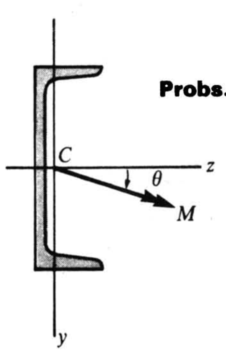
9.3-5 The cross section of a beam is in the form of an equilateral triangle with sides of length b (see figure). The beam is subjected to a bending couple M having its vector at an angle θ to the z axis. (a) Derive formulas for the stresses σ_a and σ_b at points A and B, respectively, in terms of M, b, and θ. (b) Plot a graph of the maximum tensile stress σ_{max} as a function of the angle θ for θ varying from 0 to 90°. Let the ordinate of the graph be the nondimensional quantity $\sigma_{max}b^3/32M$.

***9.3-6** A beam of semicircular cross section of radius r is subjected to a bending couple M having its vector at an angle θ to the z axis (see figure). Determine the maximum tensile stress σ_{max} in the beam for $\theta = 0$, 45°, and 90°.

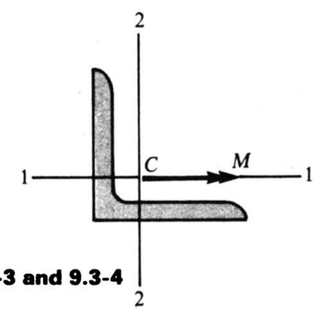
9.3-7 An angle section with unequal legs is subjected to a bending moment M having its vector along axis 1-1, as shown in the figure. Calculate the maximum tensile stress σ_t and the maximum compressive stress σ_c in the beam if the angle is an L $8 \times 6 \times 1$ section and $M =$ 25 in.-k.

9.3-8 Solve the preceding problem for an L $7 \times 4 \times \frac{1}{2}$ angle section with $M = 15$ in.-k.

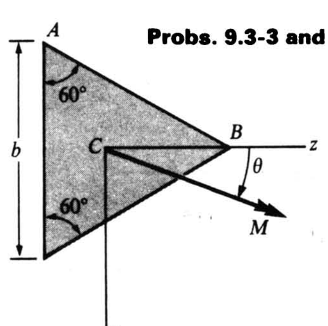
***9.3-9** A beam of Z-section is subjected to a bending moment M acting in the 2-2 plane, as shown in the figure. Calculate the maximum tensile stress σ_t and the maximum compressive stress σ_c if the moment $M = 4$ kN·m and the dimensions are $b = 90$ mm, $h = 180$ mm, and $t = 15$ mm.



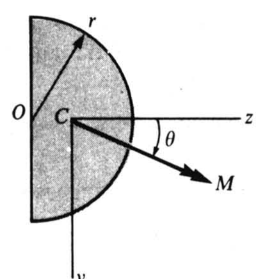
Probs. 9.3-1 and 9.3-2



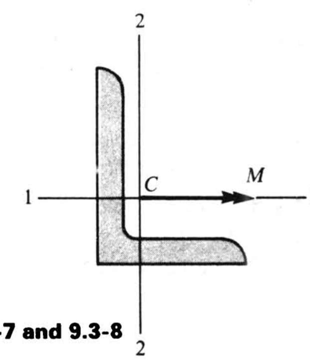
Probs. 9.3-3 and 9.3-4



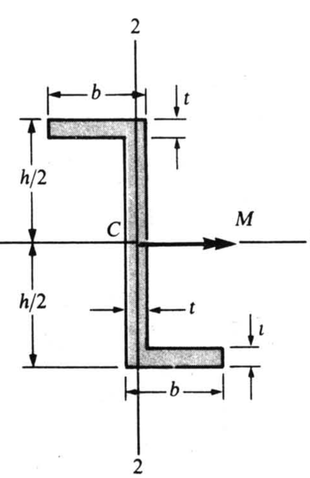
Prob. 9.3-5



Prob. 9.3-6



Probs. 9.3-7 and 9.3-8



Prob. 9.3-9

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