CHAPTER 1 Introduction

1.1 INTRODUCTION

The concept of sets is fundamental in all branches of mathematics. It was developed by German mathematician George Cantor. This chapter introduces the notations and terminology of set theory. Classical set theory, also termed as crisp set theory, is fundamental to the study of pure mathematics.

1.2 NUMBER SYSTEM

The number system plays a key role in mathematics. The real number system $\boldsymbol{R}$ is one of the most important and beautiful mathematical system. There are different ways of introducing the real number system, but the most common way is to start with Peano's Axioms for the natural numbers. The axioms for natural numbers, discovered by the Italian Mathematician Peano are:

(i) 1 is a natural number.
(ii) Each natural number n has a successor $(n+1)$.
(iii) Two natural numbers are equal if their successors are equal.
(iv) Except 1, each natural number is a successor of natural number.
(v) Any set of natural numbers which contains 1 and the successor of every natural number $(k+1)$ whenever it contains k in the set $\boldsymbol{N}$ of natural numbers.

Remarks

- Axiom (v) is commonly known as the axiom of induction or principle of finite induction.
- The above axioms completely define the set of natural numbers.

Definition: *The numbers* 1, 2, 3, … *are called natural numbers*. We represent the set of natural numbers by $\boldsymbol{N}$.

i.e., $$\boldsymbol{N} = \{1,2,3, \ldots\}$$

The Peano's axioms can be used to extend the set $\boldsymbol{N}$ of natural numbers to another large system, known as the set of integers.

Definition: *The numbers* … , –3, –2, –1, 0, 1, 2, 3… *are called integers. We represent the set of integers by* $\boldsymbol{Z}$.

i.e., $$\boldsymbol{Z} = \{ \ldots -3, -2, -1, 0,1,2,3,.. \}$$

Integers can be used to define the rational numbers.

Definition: *Any number of the form p/q, where* $p,q \in \boldsymbol{Z}$, $q \neq 0$ *and p,q have no common factor* (*except* ± 1) *is called a rational number.*

The set of rational numbers is denoted by $\boldsymbol{Q}$.

$$\therefore \qquad \boldsymbol{Q} = \left\{\frac{p}{q};\ p, q \in \boldsymbol{Z}, q \neq 0\right\}$$

REMARK

➨ The set of rational numbers consists of integers and fractions.

Definition: *Any number which is not rational, is called an irrational number.*

For example, $\sqrt{2}, \sqrt{3}$ etc. It should be noted that every rational number can be expressed as a terminating or recurring decimal whereas every irrational number can be expressed as a non-terminating infinite decimal.

1.2.1 REAL NUMBER

A number which is either rational or irrational is called a real number. The set of real numbers is denoted by $\boldsymbol{R}$.

1.2.2 INTEGRAL POWERS OF A REAL NUMBER

Let $a \in \boldsymbol{R}$, and n be any positive integer then we can define $a^n = a.a.a \ldots n$ times.

In particular

$$a = a$$
$$a^2 = a.a$$
$$a^3 = a.a.a = a^2.a \quad \text{and so on.}$$

Also, if n is any negative integer, then we have $x^{-n} = (x^n)^{-1} = (x^{-1})^n$

1.2.3 POSITIVE AND NEGATIVE REAL NUMBERS

(i) A real number a is called positive, if $a > 0$ and the set of all positive real numbers, denoted by $\mathbf{R}^+$, is given by $\mathbf{R}^+ = \{x : x \in \mathbf{R}, x > 0\}$

(ii) A real number a is called negative if $a < 0$ and the set of all negative real numbers, denoted by $\mathbf{R}^-$, is given by $\mathbf{R}^- = \{x : x \in \boldsymbol{R}, x < 0\}$

1.3 INTERVAL

A subset S of $\boldsymbol{R}$ *is called an interval if* $a, b \in S, x \in \mathbf{R}$ *such that* $a < x < b$ *implies* $x \in S$. There are following four type of intervals.

(i) a ○————○ b $\Rightarrow$ $]a, b[= \{x : a < x < b\}$

(ii) a ●————● b $\Rightarrow$ $[a, b] = \{x : a \leq x \leq b\}$

(iii) a ○————● b $\Rightarrow$ $]a, b] = \{x : a < x \leq b\}$

(iv) a ●————○ b $\Rightarrow$ $[a, b[= \{x : a \leq x < b\}$

OBSERVATIONS

➨ The set $]a, b[$ in which the end points are not included, is called an open interval.

➨ The set $[a, b]$ also contains both its end points, is called a closed interval.

➨ The sets $[a, b[$ and $]a, b]$ are called half open (or half closed) intervals or semi-open (or semi-closed) interval as they contain only one end point.

Apart from the four types of intervals listed above; there are a few more types: These are

(i) $]a, \infty[= \{x : a < x\}$ (open right ray)

(ii) $[a, \infty[= \{x : a \leq x\}$ (closed right ray)

(iii) $]-\infty, b[= \{x : x < b\}$ (open left ray)

(iv) $]\ \infty, b] = \{x : x \leq b\}$ (closed left ray)

(v) $]-\infty, \infty[=$ (open interval)

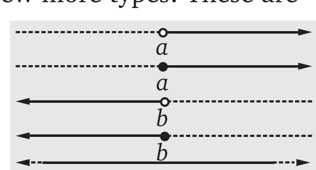

Fig. (1)

REMARKS

- If S is any interval and if c and d are two elements of S, then all numbers lying between c and d are also elements of S.
- The proper use of a bracket, for example, parenthesis' for open and square brackets for closed and end points, itself specifies the interval. As such, to emphasize the nature of an interval, we shall drop the used 'description' and shall simply express the interval by using the appropriate brackets.

1.3.1 LENGTH OF AN INTERVAL

The number $b - a$ is called length of the intervals $]a, b[$, $[a, b[$, $]a, b]$ and $[a, b]$. If the length of the interval is finite, the interval is said to be finite and if the length is infinite, then it is known as infinite interval.

1.3.2 ABSOLUTE VALUE OF A REAL NUMBER

The absolute value of a real number a denoted by $|a|$ is the real number $a, -a$ or 0 according as a is positive, negative or zero, *i.e.*,

$$|a| = \begin{cases} a & \text{if} \quad a \geq 0 \\ -a & \text{if} \quad a < 0 \end{cases}$$

From the above definition, it is clear that

(i) $|a| = \max\{a, -a\}$ (ii) $-|a| = \min\{a, -a\}$ (iii) $|a| \geq a \geq -|a|$

1.3.3 SOME USEFUL RESULTS

(i) $|xy| = |x| . |y|$ (ii) $|x+y| \leq |x| + |y|$

(iii) $|x - y| \geq ||x| - |y||$ (iv) $|x - y| \leq |x| + |y|$

(v) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ (vi) If $\in > 0$, then $|x - y| < \varepsilon \Leftrightarrow y - \in < x < y + \in$

1.4 CONCEPT OF SETS

The theory of sets is one of the most important tools of pure mathematics. Pure mathematics is the study of sets equipped with assigned structures, known as mathematical systems. In this section, we shall study some fundamental concept of set theory.

Definition: '*A set is a well defined collection of distinct objects*'.

The objects of a set are called the elements or members of that set and their membership is defined by certain conditions.

The sets are usually denoted by the capital letters of English alphabets: Say *A,B,C,... , X, Y, Z*.

For example :

(i) The collection of the letters *a, b, c, d,..*

(ii) The collection of all natural numbers denoted by ***N***.

(iii) The students of M.Sc., Mathematics in C.C.S. University, Meerut.

(iv) The collection of vowels in English alphabet. This set containing only five elements, namely *a, e, i, o, u*.

(v) The collection of all states in Indian union.

If S is a set, an object a in the collection S is called an element of S. This fact is expressed in symbol as $a \in S$ (read as a is in S or a belongs to S). If a is not in S, we write $a \notin S$.

For example, $4 \in \boldsymbol{R}$, the set of real numbers, but $\sqrt{-2} \notin \boldsymbol{R}$.

Here, Greek letter $\in$ denotes 'belongs to'. It is the abbreviation of the Greek word meaning 'is'.

REMARKS

- By the term 'well defined' we mean that we are given a collection of objects, with certain definite property, so that we are able to determine whether a given object belongs to our collection or not. Thus, every collection of objects is not a set.
- Set and aggregate both have the same meaning.
- The elements of a set must be distinguished from one another. The collection of sand particles does not form a set.
- The collection of rich persons of a city is not a set. However the collection of those persons of city whose wealth exceeds, a fixed amount, say rupees fifty thousands, is a set.
- The order is not preserved in case of a set, whereas order is necessarily preserved in case of sequence. That is to say, each of the sets {1,2,3}, {3,2,1}, {1,3,2} denotes the same sets.
- The repetition of an element does not change the nature of a set, i.e., each of the sets {1,2,3}, {1,2,2,3}, {1,3,3,2} denotes the same sets.

1.4.1 REPRESENTATION OF A SET

There are two ways of representing a set:

(i) Roster or tabulation method
(ii) Set-builder or rule method

Roster Method: In this method, the elements of the set are listed within brackets, and separated by comma.

For example:

(i) A= {1,2,3,4,5,6}
(ii) The set of vowels of English alphabet may be represent as {a, e, i, o, u}.
(iii) The set of a natural numbers from 1 to 100 may be written as $\boldsymbol{N}= \{1,2,3, \ldots, 100\}$. We use three dots in the middle to include the missing elements.
(iv) The set of positive integers, which is a non-ending set may be written as $\boldsymbol{Z}^+=\{1,2,3,4,5, \ldots\}$. The three dots in the end means that the elements continue in the same manner.
(v) The set of prime number is written as P = {2,3,5,7,11,13,17,19….}

Set-Builder Method: In this method, we first try to find a property which characterizes the elements of a set, that is, a property P, which all the elements of the set possess and which no other objects possess. Then, we describe the set as $\{x : x \text{ has property } P\}$.

This is to be read as "the set of all x such that x has property P".

For example:

(i) The set of all integers can be written as $\boldsymbol{Z}= \{x : x \text{ is an integer}\}$
(ii) The set $A = \{1, 2, 3,4, 5\}$ can be written as $\boldsymbol{A}= \{x \in \boldsymbol{N} : x \leq 5\}$.
(iii) The set of complex numbers can be written as $\boldsymbol{C} = \{a+ib : a, b \in \boldsymbol{R}\}$
(iv) The set A = {1,8,27, ….} can be written as $A = \{x^3 : x \in \boldsymbol{Z}^+\}$.

Solved Examples

EXAMPLE 1. ***Use the Roster method to identify each set:***

(a) The set of possible integers greater than 8 ***and less than*** 14***.***
(b) The set of numbers whose elements are the first five positive odd integers.

(c) The set of even positive integers.
(d) The set of even positive integers that are divisible by 10.
(e) The set of all vowels in English alphabets which precedes r.

SOLUTION. (a) {9, 10, 11, 12, 13} (b) {1,3,5,7,9}
(c) {2, 4, 6, 8, 10 …} (d) {10,20,30,40,50 …}
(e) {a, e, i, o}

EXAMPLE 2. ***Use the set-builder method to identify the following sets :***

(a) $\boldsymbol{A = \{1,3,5,7,9,\ldots\}}$ ***(b)*** $\boldsymbol{B = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right\}}$

(c) $\boldsymbol{C = \{0,1, 2, 3, \ldots.\}}$ ***(d)*** $\boldsymbol{D = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}}$

SOLUTION. (a) The set of odd positive integers.
(b) Here, elements of the set B are the reciprocals of the squares of the natural numbers.
So, the set $B = \left\{\frac{1}{n^2} : n \in \mathbf{N}\right\}$
(c) The set of whole numbers.
(d) Here, each element in the given set has the denominator one more than the numerator. Hence,

$$D = \left\{x : x = \frac{n}{n+1} : n \in \mathbf{N}\right\}$$

EXAMPLE 3. ***Write the set*** $\boldsymbol{\left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{5}{26}, \ldots\right\}}$ ***in the set-builder form.***

SOLUTION. We observe that each element in the given set has the denominator one more than the square of the numerator. Also, the numerator begins with 1. Hence, in the set builder form, the given set can be written as

$$\left\{x : x = \frac{n}{n^2+1} : n \in \mathbf{N}\right\}$$

EXERCISE 1.1

1. Which of the following collections are sets?
(i) All mathematics students in your college.
(ii) All poor hockey players in the college.
(iii) All odd numbers less than 20.
(iv) The collection of good teachers in your college.
(v) All successful and rich people in your city.
(vi) The people in your immediate family (father, mother, sister, brother).

2. Write the members of each of following sets by the Roster method.
(i) {x : x is odd whole number less than 14}
(ii) $\{x : x^2 < 36 \text{ and } x \in \mathbf{N}\}$
(iii) {x : squares of all whole numbers less than 8}
(iv) {x : x is a prime number, $10 < x < 20$}
(v) {x : x is a composite number less than 20}
(vi) $\{x : x < x\}$

3. Rewrite the following sets using set-builder method.
(i) $A = \{2,4,6,8, \ldots\}$
(ii) $B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.\right\}$
(iii) $C = \{0, 3, 6, 9, 12, \ldots\}$
(iv) $D = \{0, 4, 6, 8, 10, \ldots\}$

4. List the elements of the following sets.
   (i) $A = \{x : x^2 \le 16 : x \in \boldsymbol{Z}\}$
   (ii) $B = \{x : 1 \le x \le 5$ and $x \in \boldsymbol{N}\}$
   (iii) $C = \{x : x \in \boldsymbol{N}$ and x is a factor of 15$\}$
   (iv) $D = \{x : x$ is a month of year having 31 days$\}$
   (v) $E = \{x : x \in \boldsymbol{Z}$ and $3x - 2 = 3\}$
   (vi) $E = \{x : x$ is an integer lying between $-1/2$ and $1/2\}$
5. Use the appropriate symbols $\in$ or $\notin$ to fill in the blanks below:
   (i) 12 … the set of all numbers dividing 84.
   (ii) K … the set of all vowels of the English alphabets.
   (iii) $\frac{1}{2}$ .. the set of natural number.
   (iv) India … the set of members of UNO.
   (v) $\sqrt{2}$ …. The set of rational number
   (vi) 15 … the set of multiples of 3.

ANSWERS

1. (i), (iii), (vi)
2. (i) {1, 3, 5, 7, 9, 11, 13} (ii) {1, 2, 3, 4, 5} (iii) {0, 1, 4, 9, 16, 25, 36, 49}
   (iv) {11, 13, 17, 19} (v) {1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18} (vi) ϕ
3. (i) $A= \{x : x = 2n : n \in \boldsymbol{N}\}$ (ii) $\{1/n : n \in \boldsymbol{N}\}$
   (iii) $\{x : x = 3n, n$ is the whole number (iv)$\{x : x = 2n, n$ is the whole number$\}$
4. (i) {–4, –3, –2, –1, 0, 1, 2, 3, 4} (ii) {1, 2, 3, 4, 5} (iii) {3, 5}
   (iv) {Jan, March, May, July, August, October, December}
5. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\in$ (v) $\notin$ (vi) $\in$

1.5 TYPES OF SETS

1.5.1 EMPTY SET

A set containing no elements is called empty set and is denoted by the symbol ϕ.

For example:

(i) $\phi = \{x : x$ is a negative integer whose square is $-1\}$

(ii) $\phi = \{x : x$ is a natural number lying between 2 and 3$\}$

(iii) ϕ = {the set of such persons, who never die}

(iv) $\phi = \{x : x$ is a real number, $x^2 < 0\}$

(v) $\phi = \{x : x$ is an even prime number greater than five$\}$

(vi) ϕ = {the set of real numbers which are solution of equation $x^2 + 1 = 0$ }

(vii) $\phi = \{x : x$ is a straight ling passing through three distinct points on a circle$\}$

REMARKS

- The empty set is also known as null set or void set.
- The Roster method, the empty set is denoted by {}.
- To describe the null set, we can use any property, which is not true for any element.
- It is wrong to use the expression 'an empty' or 'a null set' as there is one and only one empty set through, it may have many-many descriptions. We shall always call 'The empty or the null set.'
- A set consisting of at least one element is called a non-empty or non-void set.
- $\{\phi\}$ is not a null set.

1.5.2 SINGLETON SET

Set containing only one element is a singleton set. The set $\{a\}$ is a singleton set.

REMARKS

- $\{0\}$ is not a null set, since it contains 0 as its member. It is a singleton set.
- A room containing only one man is not same thing as a man. In a similar way, the singleton set $\{a\}$ is not the same thing as the element a.

1.5.3 FINITE SET

A set is said to be finite if it consists of only finite number of elements. Here, the process of counting the different elements comes to an end.

For example:

(i) Set of natural numbers less than 50.
(ii) Set of all persons in a city.
(iii) Set of English alphabets.
(iv) Set of all persons on the earth.

1.5.4 INFINITE SET

A set which is not finite, i.e., it contains infinite number of elements. Here, process of counting the different elements never comes to an end.

For example:

(i) Set of natural numbers $\boldsymbol{N} = \{1,2,3, \ldots\}$
(ii) Set of all points of plane.
(iii) Set of all even integers.
(iv) Set of rational numbers lying between two integers.

1.5.5 EQUAL SETS

Two sets are said to be equal if they contain exactly the same elements.

For example:

$A = \{x : x$ is a letter in the word 'Area'$\}$, *i.e.*, $A = \{a, r, e\}$

And $B = \{y : y$ is a letter in the word 'ear'$\}$, *i.e.*, $B = \{a, r, e\}$

Here A and B are equal sets.

1.5.6 CARDINAL NUMBER OF A SET

The number of distinct elements contained in a finite set A is called cardinal number of A and is denoted by $n(A)$.

1.5.7 EQUIVALENT SETS

Two finite sets are said to be equivalent if they have the same cardinal number.

REMARKS

- Equivalent sets are not always equal but equal sets are always equivalent.
- The number of distinct elements in a finite set is also called the order of the set. If the order of a set is zero, the set is empty.
- If the order of a set is one, the set is singleton.
- The order of an infinite set is never defined.

1.6 SUBSET

Let A and B be two sets. *The set A is said to be a subset of the set B if every element of A is also an element of B.* Symbolically, we write $A \subseteq B$.

When A is subset of B, it means that 'A is contained in B' or 'B contains A'. Here B is called superset of A and is written as $B \supset A$.

Remarks

- Every set is a subset of itself.
- Empty set is a subset of every set.
- If A is not a subset of B, we write $A \not\subseteq B$.
- An element cannot be a subset of a set, only a set can be subset of a set.

1.6.1 PROPER SUBSET

We know that for A to be a subset of B all that is needed is that every element of A is in B. It is possible that every element of B may or may not be in A. If it so happens that every element of B is also in A, then we will have $B \subset A$. Obviously, then A and B are the same set, so that we have $A \subset B$ and $B \subset A \Leftrightarrow A = B$.

If every element of A is in B, but every element of B is not in A, *i.e.*, if $A \subset B$ and $B \not\subset A$, then A is said to be a proper subset of B.

For example:

(i) $\{a, b\}$ is a proper subset of $\{a, b, c\}$.

(ii) Set of natural number $\boldsymbol{N}$ is a proper subset of set $\boldsymbol{Z}$ of integer.

Remarks

- Here, it follows that every element of A is an element of B and B contains at least one element which does not belong to A.
- If the subset is not proper, it is called **improper subset. For example**, $A \subseteq A$ and $\phi \subseteq A$ are improper subsets.

1.6.2 NUMBER OF SUBSETS OF A SET

If A is a set contains n distinct element. Let $0 < r \leq n$. If we consider those subsets of A that have r elements each, then we know that the number of ways in which r elements can be choose out of n elements is ${}^nC_r$. Therefore, the number of subsets of A having r elements each is ${}^nC_r$.

Hence, the total number of subsets of A is equal to

$$ {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = (1+1)^n = 2^n $$

For example:

(i) If a set A has one element, then it has $2^1 = 2$ subsets.

(ii) If a set A has two elements, then it has $2^2 = 4$ subsets.

Remarks

- The number of proper subsets of a set with n elements is 2^{n-1}.
- The collection of all possible subsets of a given set A is called power set. It is denoted by $P(A)$. **For example :** If A = {1,2,3} then the power set $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.
- $P(\phi) = \{\phi\}$
- The power set of any given set is always non-empty.

1.7 UNIVERSAL SET

In any discussion , we are given particular set and we consider different subsets of the given set. This given set is called Universal Set. It is denoted by U.

For Example:

(i) The universal set is of real numbers ***R***, while considering the set of natural numbers, whole numbers, integers and rational numbers.

(ii) The set of alphabets is the universal set from which the letters of any word may be chosen to form a set.

(iii) In geometry, we discuss set of lines, triangles and circles, then the universal set is the plane, in which the lines, triangles and circles lie.

REMARKS

- Universal set is a super set of each of the given sets.
- The universal set is not unique.

1.7.1 COMPLEMENT OF A SET

Let U be the universal set and the set $A \subseteq U$. Complement of set A with respect to the universal set U is the set of all those elements of U which are not the elements of A and is denoted by A' or A^c,

$$A' = \{x : x \in \text{U and } x \notin A\}$$

For example:

(i) If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$ and $A = \{1, 2, 3\}$ then $A' = \{4, 5, 6, 7, 8, 9, 11\}$.

REMARKS

- Complement of the universal set is the null set and *vice-versa*.
- $(A')' = A$
- If $A \subseteq B$, then $B' \subseteq A'$.
- $x \in A' \Leftrightarrow x \notin A$

Solved Examples

EXAMPLE 1. ***Let A = {1,2,3}, then find P(A).***

SOLUTION. Since $A = \{1, 2, 3\}$ then,

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

EXAMPLE 2. ***Let A = {a,b,c,d}, B = {a,b,c} and C = {b,d}, find all sets X such that***

(i) $X \subset B$ and $X \subset C$ (ii) $X \subset A$ and $X \not\subset B$

SOLUTION. (i) Here, we have

$$P(B) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a,c\}, \{b, c\}, \{a, b, c\}\}.$$

And $P(C) = \{\phi, \{b\}, \{d\}, \{b, d\}\}$, then $X \subset B$ and $X \subset C$ implies

$$X \in P(B) \text{ and } X \in P(C)$$

Therefore, $X = \{\phi, \{b\}\}$

(ii) Here, we have, $X \subset A$ and $X \not\subset B$, which implies that

$$X \in P(A) \text{ and } X \notin P(B)$$

Therefore $X = \{\{d\}, \{a,b,d\}, \{b,c,d\}, \{a,c,d\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,c,d\}\}$

EXAMPLE 3. ***Write down all the subsets of the following sets.***

(i) {a} (ii) {a,b} (iii) {a,b,c} (iv) ϕ

SOLUTION. (i) Let $A = \{a\}$. Since A contains only one element, therefore, the total number of subsets is $2^1 = 2$, which are given by ϕ and $\{a\}$.

(ii) Here, total number of subsets, $= 2^2 = 4$, which are given by ϕ, $\{a\}$,$\{b\}$, $\{a, b\}$

(iii) Here, total number of subsets $= 2^3 = 8$, given by

$$\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$$

(iv) since ϕ contains no element therefore the number of subsets $= 2^0 = 1$. The only subset is ϕ.

EXAMPLE 4. ***Which of the following sets are empty. Also, give the reason.***

(i) $A = \{x : x \neq x$, is a real number$\}$.

(ii) $B = \{x : x + 4 = 4\}$

(iii) $C = \{x : x^3 - 3 = 0$ and x is rational number$\}$

SOLUTION. (i) Here, $A = \{x : x \neq x, x$ is a real number$\}$. Since $x \neq x$ is not true

$\Rightarrow \quad A = \phi$

(ii) $B = \{x : x + 4 = 4\} = \{x : x = 0\} = \{0\}$

$\Rightarrow \quad B$ has one element 0, therefore $B \neq \phi$.

(iii) Since there is no rational number whose square is 3, so $x^3 - 3 = 0$ is not satisfied for any rational numbers. Therefore, C is an empty set.

EXAMPLE 5. ***Which of the following sets are finite and which are infinite.***

(i) The set of natural numbers divisible by 2.

(ii) The set of natural numbers less then 8.

(iii) The set of integers whose square is even.

(iv) The set of integers greater than –18.

(v) The set of lines passing through a point.

(vi) The set of points of a plane at a fixed distance from a given point in the plane.

(vii) The set of points common to two given parallel lines.

(viii) The set of the roots of a polynomial of n^{th} degree.

SOLUTION. (i) The given set is $\{2, 4, 6, 8, \ldots\}$. It has an infinite number of elements, therefore it is an infinite set.

(ii) The given set is $\{1,2,3,4,5,6,7\}$. It has seven elements, *i.e.*, finite number of elements. Hence, it is a finite set.

(iii) The given set is $\{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\}$. It has infinite number of elements, therefore it is an infinite set.

(iv) Here, the given set is $\{-17, -16, \ldots, 0, 1, 2 \ldots\}$. It has infinite number of elements therefore, it is an infinite set.

(v) Since infinite number of lines can pass throught a fixed point, therefore the given set is an infinite set.

(vi) Since the points in a plane at a fixed distance from a given point in the plane lie on a circle with the given point as center and the number of points on a circle is infinite. Therefore, the given set is an infinite set.

(vii) Since two parallel lines cannot meet anywhere, therefore, the set of points common to two given parallel lines is empty, therefore the given set cannot be infinite. Hence, it is a finite set.

(viii) Since, a polynomial of n^{th} degree always have atmost n roots.

Therefore, the given set is always a finite set.

EXAMPLE 6. ***Which of the following sets are equivalent?***

ϕ, {0} and {ϕ}.

SOLUTION. Since ϕ has no element. Also, {0} and {ϕ}, each contains one element namely 0 and ϕ respectively. Hence, {0} and {ϕ} are equivalent.

EXAMPLE 7. ***Which of the following sets are equal ?***

$$\boldsymbol{A = \{1,2,3\},\ B = \{2,3,4\},\ C = \{3,2,1\},\ D = \{2,3,5\}}$$

SOLUTION. Since $1 \in A$ but $1 \notin B$, therefore $A \neq B$. A and C have exactly the same element, therefore $A = C$.

Also, $1 \in C$ but $1 \notin D \Rightarrow C \neq D$

$4 \in B$ but $4 \notin C \Rightarrow B \neq C$

$4 \in B$ but $4 \notin C \Rightarrow B \neq C$

$1 \in A$ but $1 \notin D \Rightarrow A \neq D$

Hence, only A and C are equal sets.

EXERCISE 1.2

1. Fill in the blanks:

(i) A set which contains no element is called … set.

(ii) If $A = \{1,2,3\}$ and $B = \{3,2,1\}$ then they are said to be …

(iii) If $A = \{a, b, c\}$ and $B = \{c, d, e\}$ then they are said to be …

(iv) If every element of a set B is also an element of A, then B is said to be … of A.

(v) The empty set is a … of every set.

(vi) Every set is a …. of itself.

(vii) The set $\boldsymbol{Z}$ of integers is a … of set of natural numbers $\boldsymbol{N}$.

2. Which of the followings sets are equal?

(i) $A = \{1,2,3\}$

(ii) $B = \{1,2,2,3\}$

(iii) $C = (x \in \boldsymbol{R} : x^3 - 6x^2 + 11x - 6 = 0)$

3. Which of the following sets are equivalent to the set {4,7,11,17,20}?

(i) {5,1,2,3,4}

(ii) {all odd numbers less then 10}

(iii) {the months of a year of 30 days}

(iv) {all the prime numbers which lie between 10 and 25}.

4. Which of the following sets are finite and which are infinite ?

(i) $\{x \in \boldsymbol{N} : x > 10\}$

(ii) $\{x \in \boldsymbol{N} : x < 100\}$

(iii) $\{x \in \boldsymbol{R} : 1 \le x \le 2\}$

(iv) Set of vowels in English alphabets.

(v) The set of prime numbers less than 100.

(vi) The set of multiple of 8.

5. Which of the following statements are true? Give the reason.

(i) For any two sets A and B either $A \subseteq B$ or $B \subseteq A$

(ii) Every subset of a finite set is finite.

(iii) A subset of an infinite set may be finite.

(iv) Every set has a proper subset.

(v) A set containing n elements have 2^n subsets.

(vi) If $A = \{1,2,3,4,5,6\}$ and B = {whole numbers less than 6}, then $A = B$.

(vii) The empty set has no proper subset.

6. Examine which of the following sets are empty?

(i) The set of tigers in your class.

(ii) The set of triangles having three equal sides.

(iii) The set of all numbers which, when added to zero, yield sum greater than the original.

(iv) The set of odd numbers which are divisible by 2.

(v) The set of men, who never die.

7. Which of the following statements are true?

(i) If $x \in A$ and $A \subset B$, then $x \in B$

(ii) If $A \subset B$ and $B \subset C$, then $A \subset C$

(iii) If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$

(iv) If $x \in A$ and $A \not\subset B$, then $x \in B$

(v) If $A \subset B$ and $x \notin B$, then $x \notin A$

8. Are the following sets, *i.e.*, (A and B) are equal.

(i) $A = \{x : x$ is a letter of the word 'LITTLE'$\}$

$B = \{x : x$ is a letter in the word 'TITLE'$\}$

(ii) $A = \{x : x$ is a letter in the word 'FOLLOW'$\}$

$B = \{x : x$ is a letter in the word 'WOLF'$\}$

(iii) $A = \{x : x$ is a letter in the word 'LOYAL'$\}$

$B = \{x : x$ is a letter In the word 'ALLOY'$\}$

9. Write down all possible subsets of each of the following sets.

(i) $\{a\}$ (ii) $\{0,1\}$ (iii) $\{a,b, c\}$

(iv) $\{1, \{1\}\}$ (v) ϕ

10. Which of the following statements are true?

(i) $\{a, \phi\} \in \{a, \{a, \phi\}\}$

(ii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

(iii) If $A \in B$ and $B \subseteq C$, then $A \in C$

(iv) If $A \subset B$ and $B \in C$, then $A \in C$

(v) If $A \subseteq B$ and $B \in C$, then $A \subseteq C$

Answers

1. (i) empty (ii) equal (iii) equivalent (iv) subset (v) subset (vi) subset (vii) super set.

2. $A = B = C$ **3.** (i), (ii), (iv)

4. (ii), (iv), (v) are finite sets and (i), (iii), (vi) are infinite.

5. (i) F (ii) T (iii) T (iv) F (v) T (vi) F (vii) T

6. (i), (iii), (iv), (v) **7.** (i), (ii), (v)

8. (i) Equal, (ii) Equal, (iii) Equal

9. (i) ϕ, {a}; (ii) ϕ ,{0},{1},{0,1}; (iii) ϕ ,$\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}$

(iv) {1};{1},{{1}},{1,{1}}; **10.** (i) , (ii), (iii), (iv), (v)

1.8 VENN DIAGRAMS

A set can be represented by closed figures like circles, triangles, rectangles, etc. The point in the interior of the figure represents the elements of the set. Such a representations is called a Venn diagram. In Venn diagram, the universal set is usually represented by a rectangular region and its subset by closed bounded regions inside the rectangular region. For example, if A is a subset of B, *i.e.*, $A \subset B$ is shown in figure 2.

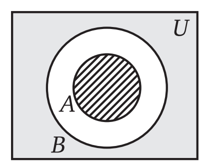

Fig. (2)

Remarks

- The diagrams drawn to represent sets are called Venn diagram or Venn-Euler diagrams, after the name of British mathematician **Venn**.
- If A and B are two sets, which are not equal, but have common elements, then to represent A and B, we draw two intersecting circles.
- Two disjoint sets are represented by two-intersecting circles.
- Venn diagrams are to be used for clarity and are no substitute for precise proof.

1.9 OPERATIONS ON SETS

1.9.1 UNION AND INTERSECTION OPERATIONS

(i) Union of two sets

Let A and B be two sets. Then Union of A and B, denoted by $A \cup B$ is the set of all those elements, which either belongs to A or B or to both A and B.

It should be noted that the common elements are to be taken only once.

Symbolically: $A \cup B = \{x : x \in A \text{ or } x \in B\}$ It is shown in the adjoining figure 3.

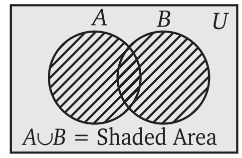

Fig. (3)

For example:

(i) Let $A = \{3,4,5,6,7\}$ and $B = \{5,6,7,8,9\}$
Then $A \cup B = \{3, 4, 5, 6, 7, 8, 9\}$

(ii) Let $A = \{x : x = 2n, n = 1, 2, 3, ...\} = \{2, 4, 6, 8, ...\}$
and $B = \{x : x = 3n, n = 1, 2, 3, ...\} = \{3, 6, 9, 12, ...\}$
Then $A \cup B = \{x : x \text{ is multiple of 2 or a multiple of 3}\}$
$= \{2, 3, 4, 6, 8, 10, 12, ...\}$

(iii) Let A = set of even natural numbers = $\{2, 4, 6, 8,...\}$
and B = set of natural numbers = $\{1, 2, 3, 4, 5,...\}$
Then $A \cup B = \{1, 2, 3, 4,...\}$

REMARKS

- $x \in (A \cup B) \Leftrightarrow x \in A$ or $x \in B$.
- $x \notin (A \cup B) \Leftrightarrow x \notin A$ and $x \notin B$
- $A \cup B = B \cup A$, *i.e.*, union of sets is commutative.
- $A \cup A' = U$ and $A \cup U = U$
- $A \cup \phi = A$
- If $A,B,C,D, ...,Z$ is a finite family of sets, then their union is denoted by $A \cup B \cup C \cup D ... \cup Z$.
- $(A \cup B) \cup C = A \cup (B \cup C)$, *i.e.*, a union of sets is associative.

(ii) Intersection of two sets

Let A and B be two sets. Then intersection of A and B, denoted by $A \cap B$ is the set of all those elements, which belongs to both A and B.

Symbolically: $A \cap B = \{x : x \in A \text{ and } x \in B\}$ It is shown in the adjoining figure 4.

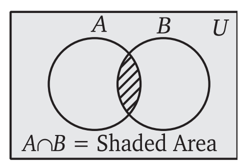

Fig. (4)

For example:

(i) Let $A=\{2, 4, 6, 8, 10\}$ and $B=\{1, 2, 3, 4, 5\}$. Then $A \cap B = \{2, 4\}$

(ii) If $A = \{x : x = 3n, n \in \mathbf{Z}\}$
and $B = \{x : x = 4n, n \in \mathbf{Z}\}$
Then $A \cap B = \{x : x \text{ is multiple of 3 and } x \text{ is a multiple of 4}\}$
$= \{x : x \text{ is multiple of 3 and 4 both}) = \{x : x = 12n, n \in \mathbf{Z}\}$

REMARKS

- $x \in (A \cap B) \Leftrightarrow x \in A$ and $x \in B$.
- $x \notin (A \cap B) \Leftrightarrow x \notin A$ or $x \notin B$
- $A = A \cap A$, *i.e.*, intersection of sets is idempotent.
- $A \cap \phi = \phi$
- $A \cap U = A$, where U is a universal set.
- $A \cap B = B \cap A$, *i.e.*, intersection of sets is commutative.
- $(A \cap B) \cap C = A \cap (B \cap C)$intersection of sets is associative.
- If $A,B,C,D, ...,Z$ is a finite family of sets, then their intersection is denoted by $A \cap B \cap C ... \cap Z$.

(iii) Distributive Property of Union and Intersection

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

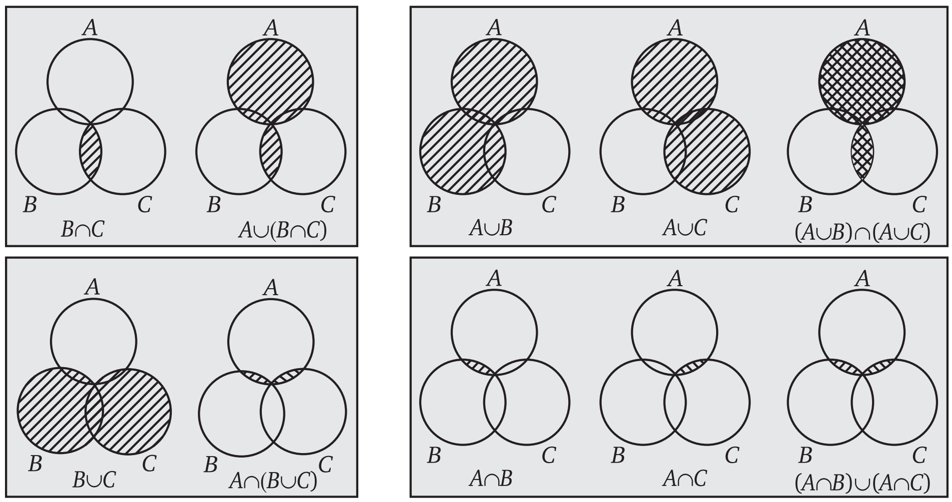

Fig. (5)

1.9.2 DISJOINT SETS

When two sets have no common elements, they are called disjoint sets. Thus, if $A \cap B = \phi$, then A and B are disjoint. It is shown in the adjoining figure 6.

For example:

(i) If $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7, 9\}$

Then, $A \cap B = \phi$

(ii) If $A =$ Boys in school

$B =$ Girls in school

Then, $A \cap B = \phi$

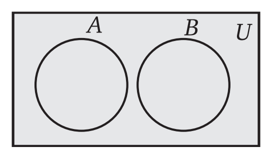

Fig. (6)

REMARKS

- If $A \cap B \neq \phi$, , then A and B are said to be intersecting or overlapping sets.
- A family of sets is said to be pairwise disjoint family of sets if and only if any two sets of this family are disjoint. For example, classes of A_2, A_3, A_5 and A_7 defined as $A_2 = \{2, 2^2, 2^3, \ldots\}; A_3 = \{3, 3^2, 3^3, \ldots\}; A_5 = \{5, 5^2, 5^3, \ldots\}$ and $A_7 = \{7, 7^2, 7^3, \ldots\}$ are pairwise disjoint.
- $\phi \cap A = \phi$, *i.e.*, null set is disjoint from every subset.

1.9.3 DIFFERENCE OF TWO SETS

If A and B are two sets, then the set of all elements which belong to A but do not belong to B is called the difference of sets A and B and is denoted by $A \sim B$. The set of all elements which belong to B but do not belong to A is called the difference of sets B and A and is denoted by $B \sim A$.

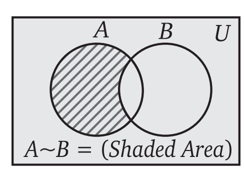

Fig. (7)

Therefore,

$$A \sim B = \{x : x \in A \text{ and } x \notin B\} = A \cap B'$$

and $B \sim A = \{x : x \notin A \text{ and } x \in B\} = B \cap A'$

For example:

(i) Let $A = \{1, 2, 3, 4, 5\}$

and $B = \{-1, 0, 1, 2\}$

Then, $A \sim B = \{3, 4, 5\}$

and $B \sim A = \{-1, 0\}$

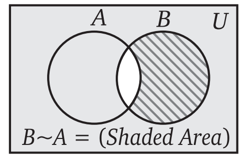

Fig. (8)

REMARKS

- $x \in (A - B) \Leftrightarrow x \in A$ and $x \notin B$.
- $x \notin (A - B) \Leftrightarrow x \notin A$ and $x \in B$
- $A \sim B \neq B \sim A$, *i.e.*, difference of two sets is not commutative.
- $A \subset B$ then $A \sim B = \phi$
- The sets $A \sim B$, $A \cap B$ and $B \sim A$ are mutually disjoint.
- $A \sim B$ is a subset of A and $B \sim A$ is a subset of B.

> Difference of a set with the universal set is known as **complementation.**

1.9.4 SYMMETRIC DIFFERENCE OF TWO SETS

If A and B are two sets, then the symmetric difference of two sets A and B is denoted by $A \Delta B$ is given by $A \Delta B = (A \sim B) \cup (B \sim A)$

Symbolically: $A \Delta B = \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$

For example:

(i) If $A = \{1,2,3,4,5,6,7,8\}$ and $B = \{1,3,5,6,7,8,9\}$

Then $A \sim B = \{2, 4\}$ and $B \sim A = \{9\}$

and $A \Delta B = \{2, 4, 9\}$

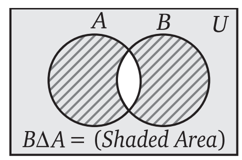

Fig. (9)

1.9.5 EQUIVALENT SETS

Two finite sets A and B are equivalent if their cardinal numbers are same , *i.e.*, $n(A) = n(B)$.

1.9.6 LAW OF EXCLUDED MIDDLE AND LAW OF CONTRADICTION

Two special properties of set operations are known as the excluded middle axioms and law of contradiction. The excluded middle axioms are very important because they are the only set operations described here that are not valid for both classical sets and fuzzy sets. Let A be any subset of universal set X. Then , we define.

(i) Axiom of the excluded middle: $A \cup A' = U$

(ii) Axiom of the contradiction: $A \cap A' = \phi$

THEOREM 1. ***(i)*** $\boldsymbol{A \cup \phi = A}$ ***(ii)*** $\boldsymbol{A \cap \phi = \phi}$ ***(iii)*** $\boldsymbol{A \cup A = A}$

(iv) $\boldsymbol{A \cap A = A}$ ***(v)*** $\boldsymbol{A \cup B = B \cup A}$ ***(vi)*** $\boldsymbol{A \cap B = B \cap A}$

PROOF. (i) Let x be an arbitrary element of $A \cap \phi$.

i.e., $x \in A \cup \phi$

Then, by definition $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

i.e., $x \in A \cup \phi \Rightarrow x \in A$ or $x \in \phi$

$\Leftrightarrow x \in A$ ($\because \phi$ is a null set $\Rightarrow x \notin \phi$)

Therefore, $A \cup \phi = A$

(ii) Let x be an arbitrary element of $A \cap \phi$.

$x \in A \cap \phi \Leftrightarrow x \in A$ and $x \in \phi$ ($\because \phi$ is a null set)

Therefore, $A \cap \phi = \phi$

(iii) Let x be an arbitrary element of $A \cup A$,

$x \in A \cup A \Leftrightarrow x \in A$ or $x \in A$ (Repeated statement)

$\Leftrightarrow x \in A$

Therefore, $A \cup A = A$

(iv) Let x be an arbitrary element of $A \cap A$,

$x \in A \cap A \Leftrightarrow x \in A$ and $x \in A$ (Repeated statement)

$\Leftrightarrow x \in A$

Therefore, $A \cap A = A$

(v) Let x be an arbitrary element of $A \cup B$,

$x \in A \cup B \Leftrightarrow x \in A$ or $x \in A$ (Writing in reverse order)

$\Leftrightarrow x \in B$ or $x \in A \Leftrightarrow x \in B \cup A$

Therefore, $A \cup B = B \cup A$

(vi) Let x be an arbitrary element of $A \cap B$

$x \in A \cap B \Leftrightarrow x \in A$ and $x \in A$ (Writing in reverse order)

$\Leftrightarrow x \in B$ and $x \in A \Leftrightarrow x \in B \cap A$

Therefore, $A \cap B = B \cap A$

THEOREM 2. ***For any three sets A, B and C***

(i) $\boldsymbol{A \cup (B \cup C) = (A \cup B) \cup C}$ ***(ii)*** $\boldsymbol{A \cap (B \cap C) = (A \cap B) \cap C}$

(iii) $\boldsymbol{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$ ***(iv)*** $\boldsymbol{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

PROOF. (i) Let x be an arbitrary element of $A \cup (B \cup C)$, then $x \in A \cup (B \cup C)$

$\Leftrightarrow$ $x \in A$ or $x \in (B \cup C)$ $\Leftrightarrow$ $x \in A$ or ($x \in B$ or $x \in C$)

$\Leftrightarrow$ ($x \in A$ or $x \in B$) or $x \in C$ (By associativity)

$\Leftrightarrow$ $x \in (A \cup B)$ or $x \in C$ $\Leftrightarrow$ $x \in$ $A \cup (B \cup C)$

Therefore, $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) Let x be an arbitrary element of $A \cap (B \cap C)$, then $x \in A \cap (B \cap C)$

$\Leftrightarrow$ $x \in A$ and $\Leftrightarrow$ $x \in (B \cap C)$ $\Leftrightarrow$ $x \in A$ and ($x \in B$ and $x \in C$)

$\Leftrightarrow$ ($x \in A$ and $x \in B$) and $x \in C$ (By associativity)

$\Leftrightarrow$ $x \in (A \cap B)$ and $x \in C \Leftrightarrow x \in (A \cap B) \cap C$

Therefore, $A \cap (B \cap C) = (A \cap B) \cap C$

(iii) Let x be an arbitrary element of $A \cup (B \cup C)$, then $x \in A \cup B(\cap C)$

$\Leftrightarrow$ $x \in A$ or $x \in (B \cap C) \Leftrightarrow$ $x \in A$ or ($x \in B$ and $x \in C$)

$\Leftrightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$) $\Leftrightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$

$\Leftrightarrow$ $x \in (A \cup B) \cap (A \cup C)$

Therefore, $A \cup (B \cap C) = (A \cup C) \cap (A \cup C)$

(iv) Let x be an arbitrary element of $A \cap (B \cup C)$, then $x \in A \cap (B \cup C)$

$\Leftrightarrow$ $x \in A$ and $x \in (B \cup C) \Leftrightarrow$ $x \in A$ and ($x \in B$ or $x \in C$)

$\Leftrightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) $\Leftrightarrow$ $x \in (A \cap B)$ or $x \in (A \cap C)$

Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

THEOREM 3. ***(i)*** $(A')' = A$ ***(ii)*** $A \cup A' = U$, ***where U is the universal set.***
(iii) $A \cap A' = \phi$ ***(iv)*** $(A \cup B)' = A' \cap B'$ ***(De' Morgan's Law)***
(v) $(A \cap B)' = A' \cup B'$ ***(De' Morgan's Law)***

PROOF. (i) Let x be an arbitrary element of $(A')'$,

$$x \in (A')' \Leftrightarrow x \notin A' \Leftrightarrow x \in A$$

Therefore, $(A')' = A$

(ii) Let x be an arbitrary element of $(A \cup A')$,

$$x \in (A \cup A') \Leftrightarrow x \in A \text{ or } x \in A' \Leftrightarrow x \in A \text{ or } x \in U - A$$
$$\Leftrightarrow x \in A \text{ or } (x \in U, x \notin A) \Leftrightarrow x \in U$$

Therefore, $A \cup A' = U$

(iii) Let x be an arbitrary element of $(A \cap A')$,

$$x \in (A \cap A') \Leftrightarrow x \in A \text{ and } x \in A' \text{ but if } x \in A \text{ then } x \notin A'$$

Therefore, $A \cap A' = \phi$

(iv) Let x be an arbitrary element of $(A \cup B)'$,

$$x \in (A \cup B)' \Leftrightarrow x \notin (A \cup B) \Leftrightarrow x \notin A \text{ and } x \notin B$$
$$\Leftrightarrow x \in A' \text{ and } x \in B' \Leftrightarrow x \in A' \cap B'$$

Therefore, $(A \cup B)' = A' \cap B'$

(v) Let x be an arbitrary element of $(A \cap B)'$,

$$x \in (A \cap B)' \Leftrightarrow x \notin (A \cap B) \Leftrightarrow x \notin A \text{ or } x \notin B$$
$$\Leftrightarrow x \in A' \text{ or } x \in B' \Leftrightarrow x \in A' \cup B'$$

Therefore, $(A \cap B)' = A' \cup B'$

Solved Examples

EXAMPLE 1. ***Show that (i)*** $A \subset (A \cup B)$, ***(ii)*** $(A \cap B) \subset A$.

SOLUTION. (i) Let $x \in A$ be arbitrary then $x \in A$ certainly but may or may not belong to B.

$$\Rightarrow \quad x \in A \cup B$$

Therefore, $x \in A \Rightarrow x \in A \cup B$ gives $A \subset A \cup B$

(ii) Let $x \in A \cap B$ where x is arbitrary

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$

In particular, $x \in A \cap B \Rightarrow x \in A$

Therefore, $(A \cap B) \subset A$

REMARK

➡ Similarly we can show that (i) $B \subset (A \cup B)$ and (ii) $A \cap B \subset B$.

EXAMPLE 2. ***Let A and B be two sets, if*** $A \cap X = B \cap X = \phi$ ***and*** $A \cup X = B \cup X$ ***for some set X, prove that A= B.***

SOLUTION. Given that $A \cup X = B \cup X$

$\Rightarrow \quad A \cap (A \cup X) = A \cap (B \cup X)$ (taking intersection by A on both sides)

$\Rightarrow \quad A = A \cap (B \cup X)$ $(\because A \cap (A \cup X) = A)$

$\Rightarrow \quad A = (A \cap B) \cup (A \cap X)$ (By distributive law)

$\Rightarrow \quad A = (A \cap B) \cup \phi \Rightarrow A = A \cap B$

$\Rightarrow \quad A \subset (A \cap B) \Rightarrow A \subset B$ ...(1)

Again consider, $A \cup X = B \cup X$

$\Rightarrow \quad B \cap (A \cup X) = B \cap (B \cup X)$ (taking intersection with B)

$\Rightarrow \quad B \cap (A \cup X) = B$

$\Rightarrow \quad (B \cap A) \cup (B \cap X) = B$ (By distributive law)

$\Rightarrow \quad (B \cap A) \cup \phi = B$ (Given $B \cap X = \phi$)

$\Rightarrow \quad (B \cap A) = B$ ($\because A \cap B = B \cap A$)

$\Rightarrow \quad A \cap B = B \quad \Rightarrow \quad B \subset A \cap B \quad \Rightarrow \quad B \subset A$ ...(2)

Hence, (1) and (2) gives $A \subset B$ and $B \subset A$.

$\Rightarrow \quad A = B$

EXAMPLE 3. ***For any two sets A and B, show that***

(i) $P(A \cap B) = P(A) \cap P(B)$, (ii) $P(A) \cup P(B) \subset P(A \cup B)$

SOLUTION : (i) Let $X \in P(A \cap B) \Rightarrow X \subset A \cap B$

$\Rightarrow \quad X \subset A$ and $x \subset B \Rightarrow X \in P(A)$ and $X \in P(B)$

$\Rightarrow \quad X \in P(A) \cap P(B)$

Therefore, $P(A \cap B) \subset P(A) \cap P(B)$ ...(1)

Now, let $X \in P(A) \cap P(B) \Rightarrow X \in P(A)$ and $X \in P(B)$

$\Rightarrow \quad X \subset A$ and $X \subset B \Rightarrow X \subset A \cap B$

$\Rightarrow \quad X \in P(A \cap B)$

Therefore, $P(A) \cap P(B) \subset P(A \cap B)$ ...(2)

From (1) and (2), we conclude that

$P(A \cap B) \subset P(A) \cap P(B)$ and $P(A) \cap P(B) \subset P(A \cap B)$ which gives

$P(A \cap B) = P(A) \cap P(B)$

(ii) Let $X \in P(A) \cup P(B) \Rightarrow X \in P(A)$ or $X \in P(B)$

$\Rightarrow \quad X \subset A$ or $x \subset B \Rightarrow X \subset A \cup B$

$\Rightarrow \quad X \in P(A \cup B)$

Therefore, $P(A) \cup P(B) \subset P(A \cup B)$

REMARK

➨ Converse of the result (ii) is not necessarily true. For example, let $A = \{1,2\}$ and $B = \{4,5,6\}$, then we find that $x = \{1,2,3,5\}$ which is a subset of $A \cup B$. Therefore, $x \in P(A \cup B)$. But $x \notin P(A)$, $x \notin P(B)$. So,

$$x \notin P(A) \cup P(B) \Rightarrow P(A \cup B) \not\subset P(A) \cup P(B)$$

1.9.7 SOME MORE RESULTS

1. If A and B are any two sets, then

(i) $A - B = A \cap B'$

(ii) $A - B = A \Leftrightarrow A \cap B = \phi$

(ii) $(A - B) \cup B = A \cup B$

(iv) $A \subset B \Leftrightarrow B' \subset A'$

(v) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

2. If A and B are any two sets, then

(i) $A - (B \cap C) = (A - B) \cup (A - C)$

(ii) $A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $A \cap (B - C) = (A \cap B) - (A \cap C)$

EXERCISE 1.3

1. Let $A = \{a, b\}$, $B = \{a, b, c\}$. Is $A \subset B$. Find $A \cup B$ and $A \cap B$.

2. If $A = \{1,2,3,4\}$, $B = \{2,4,6,8\}$, $C = \{3,4,5,6\}$ and universal set $U = \{1,2,3,4,\ldots 9\}$. Verify that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

3. If A, B , C are subsets of a set X, then show that $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

4. Find the union of the following sets:
(i) $A = \{x : x$ is an even integer$\}$,
$B = \{x : x$ is an odd integer$\}$.
(ii) $A = \{x : x$ is a multiple of 2$\}$,
$B = \{x : x$ is a multiple of 3$\}$.
(iii) $A = \{x : x$ is a rational number $\}$,
$B = \{x : x$ is an irrational number$\}$.
(iv) $A = \{x : x$ is a negative integer$\}$,
$B = \{x : x$ is a non- negative integer$\}$

5. Find the intersection of the following sets.
(i) $A = \{x : x$ is an even integer$\}$,
$B = \{x : x$ is an odd integer$\}$
(ii) $A = \{x : x$ is a rational number $\}$,
$B = \{x : x$ is an irrational number$\}$.
(iii) $A = \{x : x$ is a multiple of 5$\}$,
$B = \{x : x$ is a multiple of 2$\}$
(iv) $A = \{x : x$ is a rational number $\}$,
$B = \{x : x$ is a real number$\}$

6. If $A = \{1,2,3,4\}$, $B = \{2,4,6,8\}$ and $C = \{3,4,5,6\}$, find
(i) $(A \cup B) \cap C$
(ii) $A \cup (B \cap C)$

7. Write T for true and F for false statement.
(i) $A \in (A \cup B)$ (T/F)
(ii) $(A \cup B) \in B$ (T/F)
(iii) $(A \cap B) \in A$ (T/F)
(iv) $A \cup A = A$ and $A \cap A = A$ (T/F)
(v) If $A \cap B = \phi$, then $A \cap \phi = B$ (T/F)
(vi) If A and B are disjoint sets, then intersection of their union and intersection is the null set. (T/F)
(vii) If A is the proper subset of U, then the union of A and A' is U. (T/F)
(viii) $U' = \phi$ and $\phi' = U$ (T/F)
(ix) $(A \cup B)' = A' \cap B'$ (T/F)
(x) $A \cap A'$ is always empty (T/F)
(xi) $(A \cap B)' = A' \cup B'$ (T/F)

8. If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 6, 7, 8, 9\}$, then show that
$A \Delta B = \{2, 4, 9\}$

9. Let $A = \{x : x \in \boldsymbol{N}\}$,
$B = \{x : x = 2n : n \in \boldsymbol{N}\}$,
$C = \{x : x = 2n–1 : n \in \boldsymbol{N}\}$
and $D = \{x : x$ is a prime natural number$\}$.
Find
(i) $A \cap B$ (ii) $A \cap C$
(iii) $A \cap D$ (iv) $B \cap C$
(v) $B \cap D$ (vi) $C \cap D$

10. For any two sets A and B, prove that $P(A) = P(B)$ implies that $A = B$

11. For any two sets A and B, show that
(i) $A \cup (A \cap B) = A$
(ii) $A \cap (A \cup B) = A$
(iii) $(A \cup B) \cap (A \cap B') = A$
(iv) $A' \cup B = U \Rightarrow A \subset B$
(v) $A \subset B \Leftrightarrow B' \subset A'$
(vi) $B \subset B \subset A \Leftrightarrow A \cap B = B$

12. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5\}$ and $C = \{4, 5, 6, 7\}$. Verify that
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(iii) $A \cap (B – C) = (A \cap B) – (A \cap C)$
(iv) $A – (B \cup C) = (A – B) \cap (A – C)$
(v) $A – (B \cap C) = (A – B) \cup (A – C)$

13. Show that
(i) if a sets has only even element, then it has 2 subsets.
(ii) if $B \subset A$ and B has one element less than that of A, show that A has twice as many subset as B has.
(iii) a set with 2 element has 2^2 subsets, a set with 3 elements has 2^3 subsets and so on.

14. If $X = \{4^n – 3n – 1 : n \in \boldsymbol{N}\}$ and $Y= \{9(n –1) : n \in \boldsymbol{N}\}$, show that $X \subset Y$.

15. Show that $A – B$, $A \cap B$ and $B – A$ are pairwise disjoint.

16. Show that $A \cup B \subseteq A \cap B$ implies that $A= B$.

ANSWERS

1. (i) Yes. $\{a, b, c\}$, $\{a, b\}$;

4. (i) $A \cup B = \{x : x$ is non-zero integer$\}$ (ii) $A \cup B = \{x : x$ is a multiple of 2 or 3$\}$
(iii) $A \cup B = \{x : x$ is a real number$\}$ (iv) $A \cup B = \{x : x$ is an integer$\}$

5. (i) ϕ (ii) ϕ (iii) 10 (iv) $\{x : x$ is a rational number$\}$

6. (i) $\{3, 4, 6\}$, (ii) $\{1, 2, 3, 4, 6\}$

7. (i) T (ii) F (iii) T (iv) T (v) F (vi) T (vii) T

(viii) T (ix) T (x) T (xi) T (xii) T (xiii) T (xiv) T

9. (i) B (ii) C (iii) D (iv) ϕ (v) 2 (vi) $D - \{2\}$

1.10 SOME RESULTS ON VENN DIAGRAMS

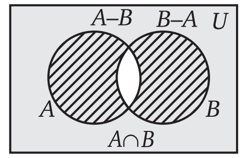

Fig. (10)

If A is a finite set, and $n(A)$ = No. of element in the set A.

The following results may be remembered for direct application :

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. $n(A \cup B) = n(A) + n(B)$, provided A and B are disjoints, *i.e.*, if $n(A \cap B) = 0$
3. $n(A \cap B') = n(A) - n(A \cap B)$
4. $n(B \cap A') = n(B) - n(A \cap B)$
5. $n(A \cup B) = n(A \cap B') + n(B \cap A') + n(A \cap B)$
6. $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
7. $n(A' \cup B') = n[(A \cap B)'] = n(U) - n(A \cap B)$
8. $n(A' \cap B') = n[(A \cup B)'] = n(U) - n(A \cup B)$
9. $n(A - B) = n(A) - n(A \cap B) \Rightarrow n(A - B) + n(A \cap B) = n(A)$
10. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Solved Examples

EXAMPLE 1. ***In a group of athletic teams in a school, 21 are in the basket ball, 26 in the hockey team and 29 in the football team. If 14 play hockey and basket ball, 12 play football and basket ball, 15 play hockey and football and 8 play all the three games. Find (i) how many players are there in all (ii) how many play football only.***

SOLUTION. Let A, B and C denote the set of players, who play basket ball, hockey and football respectively. Then, according to question, we have

$$n(A) = 21,\ n(B) = 26,\ n(C) = 29$$

$$n(A \cap B) = 14,\ n(A \cap C) = 12,\ n(B \cap C) = 15 \text{ and } n(A \cap B \cap C) = 8$$

Therefore,
$$\begin{aligned} n(A \cup B \cup C) &= [n(A) + n(B) + n(C) + n(A \cap B \cap C)] \\ &\quad - [n(A \cap B) + n(A \cap C) + n(B \cap C)] \\ &= [21 + 26 + 29 + 8] - [14 + 12 + 15] = 43 \end{aligned}$$

Hence, the total number of players is 43. Now, the number of players playing football only is $[29 - (7+8+4)] = 10$.

EXAMPLE 2. ***In a canteen, out of 123 students, 42 students buy ice-cream, 36 buy burst and 10 buy cakes, 15 students buy ice-cream and 11 buy ice-cream and buns but no cakes. Draw Venn diagram to illustrate the above information and find (i) how many students buy nothing at all (ii) how many students buy at least two items. (iii) how many students buy all three items.***

SOLUTION : Define the sets A, B and C such that

A = Set of students who buy cakes

B = Set of students who buy ice-cream

C = Set of students who buy buns

According to question, we have,

$$n(A) = 10;\ n(B) = 42;\ n(C) = 36;\ n(B \cap C) = 15;$$

$$n(A \cap B) = 10;\ \ n[(A \cap C) - B] = 4;$$

$$n[(B \cap C) - A] = 11 \text{ and } n[A - B \cup C] = 10$$

Now we have $n(B \cup C) = n(B) + n(C) - n(B \cap C)$

$$= 42 + 36 - 15 = 63$$

$$n(B \cup C) - n(B) = 63 - 42 = 21$$

and $\quad n(B \cup C) - n(C) = 63 - 36 = 27$

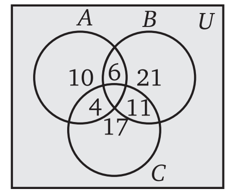

Fig. (11)

The above distribution of the students can be illustrated by Venn diagram (Figure 11).

Now, total number of students buying something.

$$= 10+6+21+4+4+11+17 = 73$$

(i) Number of students who did not buy anything = 123 – 73 = 50

(ii) Number of students buying at least two items = 6+4+4+11 = 25

and (iii) Number of students buying all three items = 4

EXERCISE 1.4

1. Out of 80 students who secured first class marks in Mathematics or in Physics, 50 obtained first class marks in Mathematics, 10 in both Physics and Mathematics. How many students secured first class marks in Physics only?
2. The Mathematics club in a school held an open house on three afternoons 115, 110 and 135 students attended both the first, second and third afternoons respectively. 25 attended just the first, 30 attended both the first and second days, 80 attended both the first and third days, and 60 attended both the second and third days. How many attended (i) all three days (ii) just the second day (iii) just the third day?
3. In a school of 250 pupils, 100 are girls, and 200 pupils stay at school for lunch. If 40 girls go home for lunch. Find the number of boys who go home for lunch.
4. In a class of 150 students, the following results were obtained in a certain examination. 45 students failed in Maths; 50 students failed in Physics, 48 students failed in Chemistry, 35 failed in both Maths and Chemistry, 25 failed in the three subject. Find the number of students who have failed in at least one subject.

ANSWERS

1. 30 **2.** 20, 30, 15 **3.** 10 **4.** 71

1.11 ORDERED PAIR

> Ordered pair may have the same first and second components, *i.e.*, two elements of an ordered pair need not be distinct.

Sometimes, there are situations in which order is very important. Some results may be affected by order and other are not.

Definition: *An ordered pair is a pair of entries whose components occur in a specific order. It is written by listing the two components in the specific order, separating them by a comma and enclosing the pair in parentheses.*

Symbolically: If A and B are two sets, then by ordered pair of elements, we must mean a pair (a,b): $a \in A$, $b \in B$ in that order.

REMARKS

- It may be noted that (a, b) is not the same as $\{a, b\}$. The former denotes an ordered pair whereas the latter denotes a set.
- $(a, b) \neq (b, a)$ unless $a = b$.
- Two ordered pairs are said to be equal when both the first components are equal and their second components are also equal.

1.11.1 CARTESIAN PRODUCT OF TWO SETS

The set of all ordered pairs of elements (a,b), $a \in A$, $b \in B$ is called the cartesian product of two sets A and B. It is denoted by $A \times B$.

Symbolically: $A \times B = \{(a, b) : a \in A, b \in B\}$

For example :

If $A = \{2, 3\}$ *and* $B = \{4,5,6\}$, then

$$A \times B = \{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$$

REMARKS

- $A \times B = \phi \Leftrightarrow A = \phi$ or $B = \phi$
- If A and B are finite sets, then $n(A \times B) = n(A).\, n(B)$
- If either A or B are infinite sets, then $A \times B$ is an infinite set.

1.11.2 ORDERED TRIPLET

If A, B, C are three sets, then by ordered triple product of elements, we mean a triplet (a,b,c) : $a \in A$, $b \in B$, $c \in C$ in that order.

This is also called **ordered 3-tuple.**

The set of all ordered triplets (a, b, c): $a \in A$, $b \in B$, $c \in C$ is also called the cartesian triple product of three sets A, B and C and is denoted by $(A \times B \times C)$

Symbolically: $A \times B \times C = \{(a, b, c): a \in A, b \in B, c \in C\}$

REMARK

- In general, the cartesian product on n sets $A_1, A_2, \ldots, A_n$ is a ordered n tuples $(a_1, a_2, \ldots, a_n)$, where $a_1 \in A_1$, $a_2 \in A_2$, …, $a_n \in A_n$. It is denoted by $A_1 \times A_2 \ldots \times A_n$ or briefly by $\prod_{i=1}^{n} A_i$ where Π stands for the product.

Solved Examples

EXAMPLE 1. ***If $A = \{1, 2\}$ and $B = \{a, b, c\}$, find the value of $A \times B$, $B \times A$, $A \times A$, $B \times B$.***

SOLUTION. We have $A = \{1, 2\}$ and $B = \{a, b, c\}$.

Therefore,

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$
$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2),\}$$
$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$
$$B \times B = \{(a, a), (a, b), (a, c), (b, a)\ (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

EXAMPLE 2. ***If $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $C = \{-1, -2\}$, find $A \times B$, $B \times A$ and $C\times(B \cup C)$.***

SOLUTION. Given that $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $C = \{-1, -2\}$.

Therefore,

$$A\times B = \{(1, a), (1, b), (1, c), (1, d), (2, a), (2, b), (2, c), (2, d), (3, a), (3, b), (3, c), (3, d)\}$$

$$B \times A = \{(a, 1), (b, 1), (c, 1), (d, 1), (a, 2), (b, 2), (c, 2), (d, 2), (a, 3), (b, 3), (c, 3), (d, 3)\}$$

Also, $B \cup C = \{a, b, c, d, -1, -2\}$

Therefore,

$$C \times (B \cup C) = \{(-1, a), (-1, b), (-1, c), (-1, d), (-1, -1), (-1, -2), (-2, a), (-2, b), (-2, c), (-2, d), (-2, -1), (-2, -2)\}$$

EXAMPLE 3. ***Find the values of a and b if $(4a-2, b+4) = (2a, 4)$.***

SOLUTION. Since we know that two ordered pairs (a_1, b_1) and (a_2, b_2) are said to be equal if $a_1 = a_2$ and $b_1 = b_2$. Therefore, for the equality of two given ordered pairs, we have

$$4a - 2 = 2a \text{ and } b + 4 = 4$$

Therefore, $4a - 2a = 2 \quad \Rightarrow \quad a = 1$ and $b + 4 = 4 \Rightarrow b = 0$

EXAMPLE 4. ***If $A = \{1, 2, 3, 4\}$ and $B = \{4, 5\}$, represent $A \times B$, $B \times A$ and $B \times B$ pictorially and find their values.***

SOLUTION. Given $A= \{1, 2, 3, 4\}$ and $B = \{4, 5\}$

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)\}$$

$$B \times A = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (4, 4), (5, 4)\}$$

And $B \times B = \{(4, 4), (4, 5), (5, 4), (5, 5)\}$

Pictorially, $A \times B$, $B \times B$ and $B \times A$ can be represented as shown in figure 12.

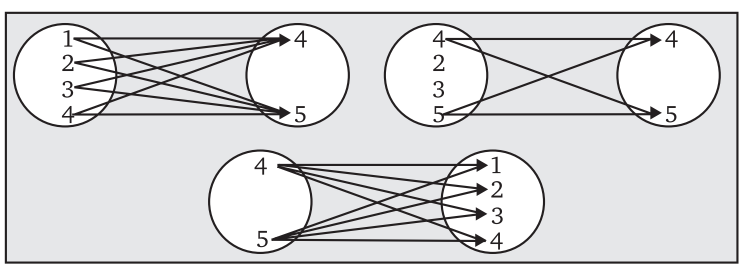

Fig. (12)

EXAMPLE 5. ***Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 7, 9\}$. Determine (i) $A\times B$, (ii) $B\times A$. Also represent $A\times B$ and $B\times A$ graphically.***

SOLUTION. (i) Given $A = \{1, 2, 3, 4\}$ and $B = \{5, 7, 9\}$. Then,

$$A \times B = \{(1, 5), (1, 7), (1, 9), (2, 5), (2, 7), (2, 9)\ (3, 5), (3, 7), (3, 9), (4, 5), (4, 7), (4, 9)\}$$

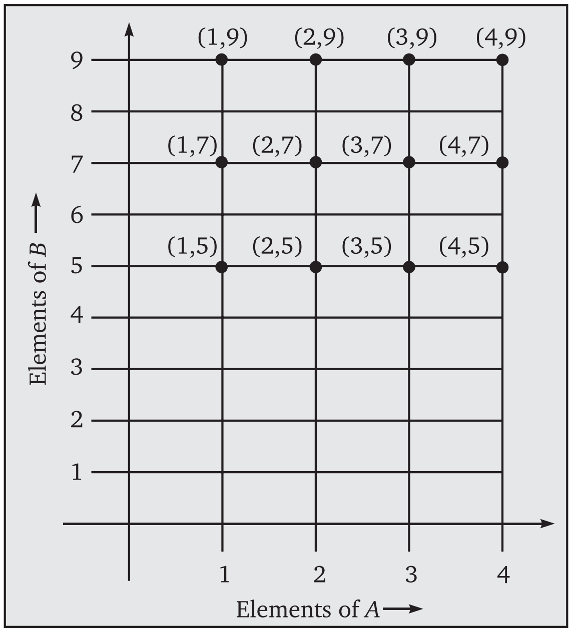

Fig. (13) : $A \times B$

Graphically, it can be represented as shown in Figure 13.

Now, $B{\times}A = \{(5, 1), (5, 2), (5, 3)\ (5, 4)\ (7, 1), (7, 2), (7, 3)\ (7, 4)\ (9, 1), (9, 2), (9\ 3)\ (9, 4)\}$

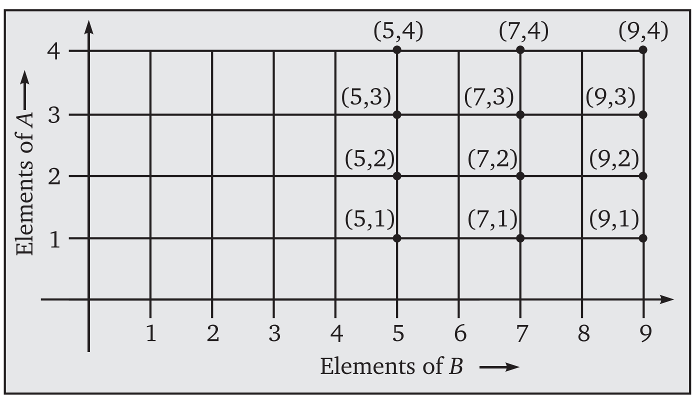

Fig. (14) : $B \times A$

Graphically, it can be represented as shown in Figure 14.

THEOREM 1. ***For any three subsets A, B and C , we have.***

(i) $A \times (B \cap C) = (A{\times}B) \cap (A{\times}C)$ (ii) $A \times (B \cup C) = (A{\times}B) \cup (A \times C)$

PROOF. (i) If $(x, y) \in A \times (B \cap C)$

$\Rightarrow$ Then, $x \in A$ and $y \in (B \cap C)$

$\Rightarrow$ $x \in A$ and $y \in B$ and $y \in C \Rightarrow x \in A, y \in B$ and $x \in A, y \in C$

$\Rightarrow$ $(x, y) \in A{\times} B$ and $(x, y) \in (A \times C) \Rightarrow (x, y) \in (A{\times}B) \cap (A{\times}C)$

But (x, y) is arbitrary, therefore

$$A{\times} (B \cap C) \subset (A{\times}B)\ (A{\times}C) \qquad \ldots (1)$$

Conversely ,

If $(x, y) \in (A \times B) \cap (A \times C)$

Then, $(x, y) \in A \times B$ and $(x, y) \in A \times C$

$\Rightarrow \quad x \in A, y \in B$ and $x \in A, y \in C \Rightarrow x \in A, y \in B$ and $y \in C$

$\Rightarrow \quad x \in A$ and $y \in (B \cap C) \quad \Rightarrow (x, y) \in A \times (B \cap C)$

But (x, y) is arbitrary, therefore

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C) \qquad \dots (2)$$

From (1) and (2), we conclude that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

(ii) $(x, y) \in A \times (B \cup C)$

Then, $x \in A$ and $y \in (B \cup C)$

$\Rightarrow \quad x \in A$ and $y \in B$ or $y \in C$

$\Rightarrow \quad (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow \quad \{(x, y) \in (A \times B)\}$ or $\{(x, y) \in (A \times C)\}$

$\Rightarrow \quad (x, y) \in (A \times B) \cup (A \times C)$

Since (x, y) is arbitrary, therefore

$$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C) \qquad \dots (1)$$

Conversely,

If $\quad (x, y) \in (A \times B) \cup (A \times C)$

Then, $\quad (x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\Rightarrow \quad (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $(y \in B$ or $y \in C) \Rightarrow (x, y) \in A \times (B \cup C)$

But (x, y) is arbitrary, therefore

$$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C) \qquad \dots (2)$$

From (1) and (2), we conclude that

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

THEOREM 2. ***For any sets A, B, C, D we have*** $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

PROOF. If $(a, b) \in (A \times B) \cap (C \times D)$, then

$\Rightarrow \quad (a, b) \in (A \times B)$ and $(a, b) \in (C \times D)$

$\Rightarrow \quad (a \in A$ and $b \in B)$ and $(a \in C$ and $b \in D)$

$\Rightarrow \quad (a \in A$ and $a \in C)$ and $(b \in B$ and $b \in D)$

$\Rightarrow \quad a \in (A \cap C)$ and $b \in (B \cap D) \Rightarrow (a, b) \in (A \cap C) \times (B \cap D)$

Since (a, b) is arbitrary, therefore

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D) \qquad \dots (1)$$

Now, let $(a, b) \in (A \cap C) \times (B \cap D)$

$\Rightarrow \quad a \in (A \cap C)$ and $b \in (B \cap D) \Rightarrow (a \in A$ and $a \in C)$ and $(b \in B$ and $b \in D)$

$\Rightarrow \quad (a \in A$ and $b \in B)$ and $(a \in C$ and $b \in D)$

$\Rightarrow \quad (a, b) \in (A \times B) \cap (C \times D)$

Since, (a, b) is arbitrary, therefore

$$(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D) \qquad \dots (2)$$

From (1) and (2), we conclude that

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

REMARKS

- $(A\times B) \cap (B\times A) = (A \cap B)\times (B \cap A)$
- $A \times (B' \cup C')' = A \times (B \cap C) = (A \times B) \cap (A\times C)$
- $A \times (B'\cap C')' = A \times (B \cup C) = (A \times B) \cup (A \times C)$

THEOREM 3. ***If A and B are two non-empty sets having n elements in common, then $A\times B$ and $B\times A$ have n^2 elements in common.***

PROOF. We know that $(A\times B) \cap (C\times D) = (A \cap C) \times (B \cap D)$

$$(A\times B) \cap (B\times A) = (A \cap B) \times(B \cap A)$$
$$(A\times B) \cap (B\times A) = (A \cap B) \times(A \cap B)$$

Since $(A\times B)$ has n elements, therefore $(A \cap B)\times(B \cap A)$ has n^2 elements.

$$(A\times B) \cap (B \times A) = (A \cap B) \times (B \cap A) \text{ has } n^2 \text{ elements.}$$

Hence, $(A\times B)$ and $(B \times A)$ have n^2 elements in common.

REMARKS

- For any three sets, $A,B,C,$ we have $A \times (B - C) = (A \times B) - (A \times C)$
- If A and B are any two non-empty sets, then $A \times B = B \times A$ iff $A = B$.
- If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
- If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set C.
- If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.
- $A \times B = A \times C \Rightarrow B = C$

EXERCISE 1.5

1. If $A= \{a, b, c\}$, $B = \{d\}$, $C = \{2\}$, then verify
   (i) $A\times(B \cup C) = (A\times B) \cup (A\times C)$
   (ii) $A\times (B \cap C) = (A\times B) \cap (A\times C)$
   (iii) $A \times (B - C) = (A \times B) - (A \times C)$
   (iv) $(A \cap B)\times C = (A\times C) \cap (B\times C)$
2. If $A = \{2, 3\}$, $B = \{1, 2, 3\}$, $C = \{2, 3, 4\}$, show that $A\times A = (B \times B) \cap (C \times C)$.
3. If $A= \{1, 2, 3\}$, $B = \{4, 5\}$ and $C = \{1, 2, 3, 4, 5\}$, then show that $(C\times B) - (A\times B) = B\times B$.
4. The ordered pairs (2,7), (4, 8) and (5, 9) and among nine elements of the set $A \times B$. Determine the other six elements of $A \times B$.
5. Let $A = \{2, 3, 5, 7\}$, $B = \{1, 12, 13, 15\}$. How many elements are there in $A \times B$? In $B \times A$? Is $A \times B = B \times A$? Is $n(A \times B) = n (B \times A)$?
6. Let A and B be two sets. Show that the sets $A \times B$ and $B \times A$ have an element in common if and only if the sets A and B have an element in common.
7. Some elements of $A \times B$ are (a, x), (a, y), (d, z). If $A : \{a, b,c\ d\}$, find the remaining elements of $A\times B$ such that $n(A\times B)$ is least.
8. If A and B are two sets having 3 elements in common. If n(A) = 5, n(B) =4, find $n(A\times B)$ and $n\{(A\times B) \cap (B\times A)\}$.
9. The ordered pairs (1, 1), (2, 2) and (3, 3) are among the elements in the set $A\times B$. If A and B have elements each, how many elements in all does the set $A\times B$ have? Also find the remaining elements.
10. If A and B are two sets such that $n(A)=3$ and $n(B) =2$. If $(x, 1)$, $(y, 2)$, $(z, 1)$ are in $A\times B$, find A and B, where x, y, z are distinct.
11. Write 'T' for true and 'F' for false statement:
    (a) If $A = (a, b)$ and $B = (b, a)$, then $A \times B = \{(a, b)\ (b, a)\}$ **(T/F)**
    (b) $\{(a, x), (a, y), (b, x), (b, y)\}$ is product set. **(T/F)**
    (c) If $n(A)= x$ and $n(B) = y$ and $A \cap B =\phi$, then $n\ (A\times B) = xy$ **(T/F)**
    (d) If A and B are non-empty sets, then $A\times B$ is a non-empty set of ordered pairs (x, y) such that $x\in A$ and $y\in A$. **(T/F)**

12. (a) If $A = \{1, 2, 3\}$, $B = \{4, 5\}$ and $C = \{1, 2, 3, 4, 5\}$. Find
(i) $A \times B$, (ii) $C \times B$, (iii) $B \times B$

(b) If $A = \{1, 2, 3, 4\}$ and $B = \{5, 7, 9\}$, find $(A \times B) \cap (A \cap B)$.

ANSWERS

4. (2, 8), (2, 9), (4, 7), (4, 9), (5, 7), (5, 8) **5.** 16, 16, No, yes

7. $(a, y), (a, 2), (b, x), (b, y), (b, z), (c, x), (c, z), (d, x), (d, y)$ **8.** 20, 9

9. 9, (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)

10. (i) $A = \{x, y, z\}$, $B = \{1, 2\}$, (ii) (a) F (b) T, (c)T (d) F

12. (a) (i) $A \times B = (1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)$
(ii) $C \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5), (5, 4), (5, 5), \}$
(iii) $B \times B = \{(4, 4), (4, 5), (5, 5)\}$ (b) ϕ

1.12 RELATION

Let us take two sets of natural numbers N_1 and N_2. We define R as a relation between them such that N_1 is a square of N_2. Then we can write $1R1$, $2R4$, $3R9$, …

In terms of ordered pair, we can write

$$R = \{(1, 1), (2, 4), (3, 9), (4, 16), \ldots\} = \{(x, y : x, y \in \boldsymbol{N} \text{ and } y = x^2\}$$

The relation from set $\boldsymbol{N}$ to $\boldsymbol{N}$ is a subset of $\boldsymbol{N} \times \boldsymbol{N}$ such that $y = x^2$.

Definition: *Let A and B be two sets. Then a relation R from A to B is a subset of A×B.*

Symbolically: R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

REMARKS

➨ If R is a relation from A to B, then A is called the domain and B the range of R.

➨ If R is a relation from a non-empty set A to a non-empty set B and if $(a, b) \in R$, then we write aRb, read as "a is related to b by the relation R." On the other hand, if $(a, b) \notin R$, we write $a\not{R}b$ and say that 'a is not related to b by the relation R'.

➨ In particular, any subset $A \times A$ defined a relation in A, known as **Binary relation.**

☛ ILLUSTRATIONS

✦ If $a, b \in \boldsymbol{N}$ and R is defined as "a is divisor of b" then R is relation on $\boldsymbol{N}$.

The subset $\boldsymbol{N} \times \boldsymbol{N}$, which corresponds to the relation R is $S = \{(n, r): n \in \boldsymbol{N}, r \in \boldsymbol{N}\}$

Here, it is clear that (1, 3), (2, 4), (3, 9) (4, 8), (4, 4), are in S, whereas (2, 3), (4, 5), (5, 6) are not in S.

✦ If R is a relation from set $A = \{1,2,3\}$ to the set $B = \{-1, -2\}$ defined by $x + y = 0$, then $R = \{(1, -1), (2, -2)\}$

Here, domain of R is $\{1, 2\}$ and Range $= \{-1, -2\}$.

✦ If $A = \{a, b, c, d, e\}$ and $B = \{f, g, h, i\}$ and let $R = \{(a, g), (a, i), (d, h), (e, f)\}$ by a relation from A to B then
Domain of $R = \{a, d, e\}$ and Range of $R = \{g, i, h, f\}$

✦ If $a, b \in \boldsymbol{R}$, the set of real numbers and $\boldsymbol{R}$ is "$|a - b|$ is a rational number" then R is a relation on $\boldsymbol{R}$. The subset S of $\boldsymbol{R} \times \boldsymbol{R}$ which corresponds to the relation is

$$S = \{(a, b + a): a \in \boldsymbol{R}, b \in \boldsymbol{Q}\}$$

It is observed that $\left(1, 2\frac{1}{2}\right), \left(\pi, 1 - \frac{1}{2}\right)$ belongs to S, while $(\sqrt{2}, \pi + \sqrt{2}) \notin S$.

✦ If $A = \{2, 3, 4\}$and $B = \{a, b, c\}$, then $R = \{(2, b), (3, c), (2, a), (4, a)\}$ being a subset of $A\times B$, is a relation from $A\times B$. Here $(2, b), (3, c), (2, a), (4, a) \in R$, so we may write $2Rb$, $3Rc$, $2Ra$, $4Ra$. But $(3, b) \notin R$ therefore, $3 R b$.

✦ If $a, b \in \boldsymbol{N}$ and R is defined by "$a - b$ is divisible by a number $n \in \boldsymbol{N}$", then R is a relation on $\boldsymbol{N}$. The subset S of $\boldsymbol{N} \times \boldsymbol{N}$ corresponding to the relation by

$$S = \{n, n + rm : n \in \boldsymbol{N}, r \in \boldsymbol{N}\}$$

Here, $m = 3, (2, 8), (5, 11) \in S$ [$\because$ 2 – 8 = 6, which is divisible by 3]

While $(3, 8) \in S$ [$\because$ 3 – 8 = 5, which is not divisible by 3]

1.12.1 TOTAL NUMBER OF RELATIONS

Let A and B be two non-empty finite sets consisting p and q elements respectively, then $A \times B$ consists of $p\,q$ ordered pairs. Therefore, total number of subset of $A \times B$ is 2^{pq}.

Remarks

➨ For a non-empty set A, $\phi \in A \times A$, therefore it is a relation on A, called **void** or **empty** relation on A.

➨ The void relation ϕ and the universal relation $A \times B$ are called trivial relation from A to B.

➨ The void and universal relation on set A respectively the smallest and the largest relation on A.

1.12.2 IDENTITY RELATION

Let A be a set. The identity relation on A is the relation $I_A = \{(x, x) : x \in A\}$ on A.

For example : If $A = \{a, b, c\}$ then the relation $I_A = \{(a, a), (b, b), (c, c)\}$is the identity relation. $R=\{(a, a), (b, b)\}$ is not an identity relation as $(c, c) \notin R$.

1.12.3 INVERSE OF A RELATION

Let A, B be two non-empty sets and R be a relation from a set A to B and let (x,y), number of the subset D of $A \times B$ corresponding to the relation R from A to B.

To the relation R from the set A to the set B, there corresponds a relation from the set B to the set A called the inverse of the relation, denoted by R^{-1} such that the subset $B \times A$ corresponding to the relation R^{-1} is $= \{(y, x): (x, y) \in D\}$.

i.e., $$yR^{-1}x \Leftrightarrow xRy$$

For example:

(i) Let $A = \{a, b, c\}$ and $B = \{1,2,3\}$ be two sets and let $R = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$ be a relation from A to B then $R^{-1} = \{(1, a), (2, a), (1, b), (2, b)\}$

(ii) If $A = \{1, 2, 3\}$, $B = \{5, 6, 7\}$ and let $R = \{(1, 5), (2, 5), (2, 7)\}$ be a relation from A to B.

Then $R^{-1} = \{(5, 1), (5, 2), (7, 2)\}$ which is a relation from B to A.

Also, Domain $(R) = \{1, 2\} =$ Range (R^{-1})

And, Range $(R) = \{5, 7\} =$ Domain (R^{-1})

(iii) The inverse of the relation "*is less than*" in $\boldsymbol{R}$ "*is greater than*".

Remark

➨ It may be noted that sometimes, the inverse of a relation coincides with the relation itself.

For example, the inverse of the relation "perpendicular to" in the set of straight lines coincides with itself.

1.13 CLASSIFICATION OF RELATIONS

1.13.1 REFLEXIVE RELATION

Let R be a relation on a set A.

"*A relation R is said to be reflexive if* $(x, x) \in R\ \forall\ x \in A$" *i.e.*, $x\,R\,x\ \forall\ x \in A$

For example :

(i) In a set of integers, a relation R defined by $x\,R\,y$ iff $x - y$ is divisible by 4, then R is a reflexive relation because $x - x = 0$ which is a divisible by 4.

(ii) The universal relation on a non-empty set A is reflexive.

(iii) The relation "is less than," *i.e.*, '$<$' in the set of rational number is not reflexive, because no member have the relation is less than to itself.

(iv) The relation "is a factor of" in the set of rational number is reflexive, since every rational number is a factor of itself.

(v) The relation "is less than or equal to." *i.e.*, $\leq$ is in the set of natural number is reflexive.

$$n \leq n\ \forall\ n \in \boldsymbol{N}$$

1.13.2 SYMMETRIC RELATION

A relation R on a set A is said to be symmetric if

$$(y, x) \in R \text{ whenever } (x, y) \in R\ \forall\ x, y \in R$$

i.e.,
$$x\,R\,y \Leftrightarrow y\,R\,x\ \forall\ x, y \in R$$

For example:

(i) Let l_1, l_2 be two lines such that l_1 is perpendicular to l_2,
i.e., $l_1 \perp l_2$. Then $l_1 \perp l_2 \Rightarrow l_2 \perp l_1$. Therefore the relation $\perp$ is symmetric.

(ii) The identity and the universal relation on a non-empty set are symmetric relations.

(iii) Consider the set $\boldsymbol{N}$ of natural numbers and the relation 'is less than'. This relation is not symmetric. Since if $2 < 3$ then $3 \nless 2$.
Let $A = \{1, 2, 3\}$ and relations R_1 and R_2 defined by
$R_1 = \{(1, 2), (1, 3), (3, 1), (2, 1)\}$ and $R_2 = \{(1, 2), (2, 3), (3, 1)\}$
Then R_1 is a symmetric relation, but R_2 is not symmetric.

1.13.3 TRANSITIVE RELATION

A relation R on a set A is said to be transitive iff $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ $\forall\ x, y, z \in A$, *i.e.*, $x\,R\,y$, $y\,R\,z \Rightarrow xRz$.

For example:

(i) Let a, b, c be three numbers such that a is a factor of b and b is a factor of c, then obviously a is a factor of c. Therefore, 'is a factor of' is a transitive relation.

(ii) If l_1, l_2, l_3 are three lines such that $l_1 \perp l_2$ and $l_2 \perp l_3$ then it is obvious that l_1 is parallel to l_3. Therefore the relation "$\perp$" is not transitive.

(iii) The identity and universal relation on a non-empty set are transitive.

(iv) Let l_1, l_2, l_3 be three straight lines, such that l_1 is parallel to l_2 and l_2 is parallel to l_3 then it is clear that l_1 is parallel to l_3. Therefore, 'is parallel to' is a transitive relation.

1.13.4 ANTI-SYMMETRIC RELATION

A relation R on a non-empty set A is said to be an anti- symmetric relation iff $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y\ \forall\ x, y \in R$

REMARKS

- The identity relation R on a set A is an anti – symmetric relation.
- If $(x, y) \in R$ and $(y, x) \notin R$, then it may be noted that $x = y$.
- The universal relation on a set A containing at least two elements is not anti – symmetric.

1.13.5 EQUIVALENCE RELATIONS

A relation R on a set E is said to be equivalence if it is

(i) Reflexive, (ii) Symmetric and (iii) Tansitive

For example :

(i) In a set of integers, a relation R is defined by $x\,R\,y$ if and only if $x - y$ is divisible by 4. Then R is an equivalence relation. Since

(a) For $x\,R\,x$, $x - x = 0$ is divisible by 4. Therefore, it is reflexive.

(b) For $x\,R\,y$. Let $x - y = 4m$ so $y - x = 4m$, which is also divisible by 4. Therefore, it is symmetric.

(c) For $x\,R\,y$, let $x - y = 4m$; for $y\,R\,z$, let $y - z = 4n$. By adding these two equations, we get $x - z = -4(m + n)$,
which is divisible by 4. Therefore it is transitive.

(ii) Let R be a relation on the set of all lines in a plane L defined by $(l_1, l_2) \in R$ if and only if line l_1 is parallel to l_2, then R is an equivalence relation because

(a) For each line $l \in L$, we have l is parallel to l.
$\Rightarrow lRl \Rightarrow R$ is reflexive.

(b) Let $l_1, l_2 \in L$ such that $(l_1, l_2) \in R$, then
$\Rightarrow (l_1, l_2) \in R \Rightarrow l_1$ is parallel to $l_2 \Rightarrow l$ is symmetric.

(c) Let $l_1, l_2, l_3 \in L$ such that (l_1, l_2) and $(l_2, l_3) \in R$, then obviously $(l_1, l_3) \in R$ because if l_1 is parallel to l_2 and l_2 is parallel to l_3, then l_3 should be parallel to l_1.

1.13.6 CONGRUENCE MODULO 'm'

Let m be an arbitrary but fixed integer. If $x - y$ is divisible by m, then two integers x and y are said to be congruence modulo m of one another.

Symbolically: $x \equiv y \pmod{m}$ is $x - y$ divisible by m.

For example: $32 \equiv 2$ (mol 3), as $32 - 2 = 30$ which is divisible by 3.

1.13.7 COMPOSITION OF RELATIONS

Let R_1 and R_2 be two relations from sets A to B and B to C respectively, then we can define a relation $R_1\,o\,R_2$ from A to C, such that $(x, z) \in R_1\,o\,R_2$ if and only if there exist $y \in Y$ such that $(x, y) \in R_1$ and $(y, z) \in R_2$.

This relation is called composition of R_1 and R_2.

REMARKS

- $R_1 o R_2 \neq R_2 o R_1$
- $(R_2 o R_1)^{-1} = R_1^{-1} o R_2^{-1}$

For example : Let A, B, C be three sets such that

$A = \{-1, -2\}$, $B = \{p, q, r\}$ and $C = \{\alpha, \beta, \gamma\}$

Also, $R_1 = \{(-1, p), (-1, r), (-2, q)\}$ is a relation from A and B *and*
$R_2 = \{(p, \alpha), (q, \beta), (r, \gamma)\}$and is a relation from set B to C.

Then $R_2 o R_1$ is a relation from A to C given by

$$R_2 o R_1 = \{(-1, \alpha), (-1, \gamma), (-2, \beta)\}$$

THEOREM 1. ***The intersection of two equivalence relations on a set is an equivalence relation.***

PROOF. Let R_1, R_2 be two equivalence relation on a set A. To show $(R_1 \cap R_2)$ also an equivalence relation.

(i) Let $a \in A$ and a is arbitrary.
Since R_1 and R_2 both are reflexive on A.
$\therefore$ $(a, a) \in R_1$ and $(a, a) \in R_2 \Rightarrow (a, a) \in R_1 \cap R_2$
Therefore, $(R_1 \cap R_2)$ is reflexive.

(ii) Let $a, b \in A$ such that $(a, b) \in R_1 \cap R_2$

$$(a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1 \text{ and } (a, b) \in R_2$$

Also, R_1 and R_2 both are symmetric on A.
Therefore, $(b, a) \in R_1$ and $(b, a) \in R_2 \Rightarrow (b, a) \in R_1 \cap R_2 \Rightarrow (R_1 \cap R_2)$ is symmetric on A.

(iii) Let $a, b, c \in$ A such that $(a, b) \in R_1 \cap R_2$, $(b, c) \in R_1 \cap R_2$
Then, $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$
$\Rightarrow$ $\{(a, b\} \in R_1$ and $(a, b) \in R_2$ and $\{(b, c) \in R_1$ and $(b, c) \in R_2\}$
$\Rightarrow$ $\{(a, b) \in R_1, (b, c) \in R_1\}$ and $\{(a, b) \in R_2, (b, c) \in R_2\}$
$\Rightarrow$ $(a, c) \in R_1$ and $(a, c) \in R_2$ [$\because$ R_1 and R_2 both are transitive]
$\Rightarrow$ $(a, c) \in R_1 \cap R_2$
Therefore, $(R_1 \cap R_2)$ is transitive on A.
From (i), (ii) and (iii), we have that $R_1 \cap R_2$ is reflexive, symmetric and transitive, and hence $R_1 \cap R_2$ is an equivalence relation.

REMARK

- The union of two equivalence relations on a set is not necessarily an equivalence relation.

THEOREM 2. ***If R is an equivalence relation, then R^{-1} is also an equivalence relation.***

PROOF. Let R be an equivalence relation on a set A. Then by definition of relation on a set, we have

$$R \subseteq A \times A \Rightarrow R^{-1} \subseteq A \times A$$

Therefore, R^{-1} is a relation on A.

Now, to show R^{-1} is an equivalence relation.

(i) Let a $\in$ A, then $(a, a) \in R$ ($\because$ R is an equivalence relation and hence reflexive)
$\Rightarrow$ $(a, a) \in R^{-1}$
Thus, $(a, a) \in R^{-1}\ \forall$ a $\in R \Rightarrow R^{-1}$ is reflexive on A.

(ii) Let $(a, b) \in R^{-1}$, then $(a, b) \in R^{-1} \Rightarrow (b, a) \in$ R
$\Rightarrow$ $(a, b) \in R$ ($\because$ R is symmetric)
$\Rightarrow$ $(b, a) \in R^{-1}$
Therefore R^{-1} is symmetric .

(iii) Let $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$ then $(a, b) \in R^{-1} \Rightarrow (b, a) \in R$
and $(b, c) \in R^{-1} \Rightarrow (c, b) \in R$
Now, $(c, b) \in R$ and $(b, a) \in R$
$(c, a) \in R$ ($\because$ R is transitive)
$(a, c) \in R^{-1}$
Therefore R^{-1} is transitive.
From (i), (ii) and (iii), we conclude that R^{-1} is an equivalence relation.

Solved Examples

EXAMPLE 1. ***Let Z be the set of integers. Define a relation R on Z such that x R y holds if and only if x – y is divisible by 5, $x \in Z$, $y \in Z$. Show that it is an equivalence relation.***

SOLUTION : (i) For each $x \in \mathbf{Z}$, $x - x$ *i.e.*, 0 is divisible by 5.

Therefore, for all $x \in \mathbf{Z}$, $x\,R\,x \Rightarrow x$ is reflexive.

(ii) Let $x\,R\,y \Rightarrow x - y$ is divisible by 5.

$\Rightarrow y - x$ is divisible by 5.

Thus $xRy = yRx$

Therefore R is symmetric.

(iii) Let us suppose xRy and yRz, then $(x - y)$ and $(y - z)$ are both divisible by 5.

Hence, 5 is also a divisor of $(x - y) + (y - z)$.

5 is a divisor of $(x - z)$.

Therefore, $xRy, yRz \Rightarrow xRz \Rightarrow R$ is transitive.

From (i), (ii) and (iii), we conclude that R is an equivalence relation.

EXAMPLE 2. ***Let N×N be the set of ordered pairs of natural numbers. Also, let R be the relation in N×N, defined by (a, b) R (c, d) if and only if a+d = b+c. Show that R is an equivalence relation.***

SOLUTION : (i) For all $(a, b) \in \mathbf{N}\times\mathbf{N}$, we have $a+b = b+a$, *i.e.,* $(a, b)\,R\,(b, a)$.

Therefore, R is reflexive.

(ii) Let $(a, b)\,R\,(c, d)$, then, by definition of R

$$(a+d) = (b+c) \text{ or } (c+b) = (d+a)$$

$(c, d)\,R\,(a, b) \Rightarrow R$ is symmetric.

(iii) Let us suppose $(a, b)\,R\,(c, d)$ and $(c, d)\,R\,(e, f)$, then

$$a + d = b + c \text{ and } c + f = d + e$$

$\Rightarrow \quad (a + d) + (c + f) = (b + c) + (d + e) \quad \Rightarrow \quad a + f = b + e$

$\Rightarrow \quad (a, b)R(e, f)$

Therefore, R is transitive.

Hence, from (i), (ii) and (iii), we conclude that R is an equivalence relation.

EXAMPLE 3. ***If R is the relation for natural number defined by x + 4y = 20. Find the domain and range of the relation R.***

SOLUTION. Let $x + 4y = 20 \quad \Rightarrow \quad y = \dfrac{20 - x}{4}$

For $x = 4$, $y = 4$ and for $x = 8$, $y = 3$.

For $x = 16$, $y = 1$ and for $x = 12$, $y = 2$

Therefore, Domain = {4, 8, 12, 16} and range = {4, 3, 2, 1}

EXAMPLE 4. ***A relation R defined on the set of integers Z, as follows***

$$(x, y) \in R \Leftrightarrow x^2 + y^2 = 25$$

Express R and R^{-1} as the sets of ordered pairs and hence find their respective domains.

SOLUTION. Since $(x, y) \in R \Leftrightarrow x^2 + y^2 = 25 \quad \Rightarrow \quad y = \pm\sqrt{25 - x^2}$

If $\quad x = 0 \quad \Rightarrow \quad y = 5.$

Therefore, $(0, 5) \in R$ and $(0, -5) \in R$

Now, $x = 3 \quad \Rightarrow \quad y = \sqrt{25-9} = \pm 4$

$(3, 4) \in R$, $(-3, 4) \in R$, $(3, -4) \in R$ and $(-3, -4) \in R$

$x = \pm 4 \quad \Rightarrow \quad y = \pm 3$

Therefore, $(4, 3) \in R$, $(-4, 3) \in R$, $(4, -3) \in R$ and $(-4, -3) \in R$

$x = \pm 5 \quad \Rightarrow \quad y = \sqrt{25-25} = 0 \quad \therefore \quad (5, 0) \in R$ and $(-5, 0) \in R$

Here, it is clear that for any other integral value of x, y is not an integer. Therefore,

$R = \{(0, 5), (0, -5), (3, 4), (-3, 4), (3, -4), (-3, -4), (4, 3), (-4, 3), (4, -3), (-4, -3), (5, 0), (-5, 0)\}$

and $R^{-1} = \{(5, 0), (-5, 0), (4, 3), (4, -3), (-4, 3), (-4, -3), (3, 4), (3, -4), (-3, 4), (-3, -4), (0, 5), (0, -5)\}$

Also, Domain $(R) = \{0, 3, -3, 4, -4, 5, -5\}$ = domain of (R^{-1}).

EXAMPLE 5. ***Consider the set A = {a, b, c}. Give an example of a relation R on A which is***

(i) reflexive and symmetric but not transitive.

(ii) symmetric and transitive, but not reflexive.

(iii) reflexive and transitive, but not symmetric.

SOLUTION. (i) Given $A = \{a, b, c\}$

Let $\quad R = \{(a, a), (a, b), (b, a), (b, c), (c, b), (b, b), (c, c)\}$ on A.

Clearly, R is reflexive and symmetric but not transitive.

(ii) Let $\quad R = \{(a, a), (a, b), (b, a), (b, b)\}$ on A.

Here, R is symmetric and transitive but not reflexive.

(iii) Let $\quad R = \{(a, a), (b, b), (c, c), (a, b)\}$ on A.

Here, R is reflexive, transitive but not symmetric.

EXAMPLE 6. ***If R is a relation in N×N, show that the relation R defined by (a, b) R (c, d) if and only if ad = bc is an equivalence relation.***

SOLUTION. (i) Since $ab = ba \; \forall \; a, b \in \mathbf{N}$.

Therefore, $(a, b) \, R \, (a, b) \forall \; a, b \in \mathbf{N} \Rightarrow R$ is reflexive.

(ii) We have $(a, b) \, R \, (c, d)$ iff $ad = bc \; \forall \; a, b, c, d \in \mathbf{N}$

Now, $(c, d) \, R \, (a, b)$ iff $cb = da \; \forall \; a, b, c, d \in \mathbf{N} \Rightarrow R$ is symmetric.

(iii) We have $(a, b) \, R \, (c, d)$ iff $ad = bc \; \forall \; a, b, c, d \in \mathbf{N}$

Therefore, $(a, b) \, R \, (c, d)$, $(c, d) \, R \, (e, f) \Rightarrow (a, b) \, R \, (e, f) \; \forall \; a, b, c, d \in \mathbf{N}$

Using $(a, d), (c, f) = (b, c)(d, e)$

$\Rightarrow \qquad (a, f) = (b, e) \;\Rightarrow\; R$ is transitive

Hence, from (i), (ii) and (iii), we conclude that R is an equivalence relation.

EXAMPLE 7. ***Let R_1 and R_2 be two relations on a set A, where A = {1, 2, 3, 5} such that***

$$R_1 = \{(1, 1), (1, 2), (1, 5), (2, 1), (2, 5)\}$$

and $$R_2 = \{(3, 3), (3, 2), (2, 3), (1, 2), (2, 1)\}$$

Then, which of the following statement is false :

(i) $R_1 \cup R_2$ ***is symmetric*** ***(ii)*** $R_1 \cap R_2$ ***is transitive***

(iii) $R_1 \cap R_2$ ***is symmetric*** ***(iv)*** $R_1 \cup R_2$ ***is transitive.***

SOLUTION. (i) As $(1, 2) \in R_1$, also $(2, 1) \in R_1$, therefore, it is symmetric and as $(1, 2) \in R_2$, also $(2, 1) \in R_2 \Rightarrow R_2$ is symmetric.

Now, $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 5), (2, 1), (2, 5), (3, 3), (3, 2), (2, 3)\}$

In $R_1 \cup R_2$, as $(1, 2) \in R_1 \cup R_2$, also $(2, 1) \in R_1 \cup R_2 \Rightarrow R_1 \cup R_2$ is symmetric

Therefore, (i) is true.

(ii) We have $R_1 \cap R_2 = \{(1, 2), (2, 1)\}$

$\Rightarrow$ (1, 1) should also belong to $R_1 \cap R_2$.

But in this case $(1, 1) \notin R_1 \cap R_2$ is not transitive.

Therefore, (ii) is false.

(iii) We have, $R_1 \cap R_2 = \{(1, 2), (2,1)\}$

$(1, 2) \in R_1 \cap R_2$ and also $(2, 1) \in R_1 \cap R_2$.

Therefore, (iii) is true.

(iv) In $R_1 \cup R_2$, $(1, 2) \in R_1 \cup R_2$

and $(2, 5) \in R_1 \cup R_2$, also $(1, 5) \in R_1 \cup R_2$

$\Rightarrow$ $R_1 \cup R_2$ is transitive

Therefore, (iv) is true.

EXAMPLE 8. ***If A be the set of all triangles in a plane and R = $\{(a, b) : \Delta a = \Delta b\}$, i.e., aRb $\Leftrightarrow$ Area of triangle a = Area of triangle b, then show that R is an equivalence relation.***

SOLUTION. (i) Since, for all $a \in A$ we have $\Delta a = \Delta a$

Therefore, $aRa \Rightarrow R$ is reflexive.

(ii) For any $a, b \in A$, we have $(a, b) \in R \quad \Rightarrow \Delta a = \Delta b$

$\Rightarrow \quad \Delta b = \Delta a \Rightarrow \quad (b, a) \in R$

Therefore, $(b, a) \in R$, *i.e.*, $bRa \Rightarrow R$ is symmetric.

(iii) For all $a, b, c \in A$, we have $(a, b) \in R$, $(b, c) \in R$

$\Delta a = \Delta b$ and $\Delta b = \Delta c \quad \Rightarrow \quad \Delta a = \Delta c \quad \Rightarrow \quad (a, c) \in R$

Therefore, R is transitive.

Hence, from (i), (ii) and (iii), we conclude that R is an equivalence relation.

EXAMPLE 9. ***If Z be a set of non-zero integers and a relation R defined by xRy $\Leftrightarrow$ $x^y = y^x$ $\forall$ x, y $\in$ Z, then show that R is not an equivalence relation on Z.***

SOLUTION. (i) Let $x \in \mathbf{Z}$, then $x^x = x^x$, $\forall x \in \mathbf{Z}$

$\Rightarrow \quad xRx, \forall x \in \mathbf{Z}$

Therefore, R is reflexive.

(ii) Let $x, y \in \mathbf{Z}$, such that xRy, *i.e.*, $x^y = y^x$

$\Rightarrow \quad x^y = y^x \Rightarrow y^x = x^y$

Therefore, $xRy \Rightarrow yRx, \forall x, y \in \mathbf{Z}$

$\Rightarrow \quad R$ is symmetric.

(iii) Let $x, y, z \in \mathbf{Z}$ such that xRy and yRz

i.e., $\quad x^y = y^x$ and $y^z = z^y$ which does not give $x^z = z^x$

$\Rightarrow \quad R$ is not transitive.

Hence, we conclude that R is not an equivalence relation.

EXAMPLE 10. ***Let $A = R \times R$ (R is the set of real numbers) and define the following relation on A : (a, b) R (c, d) iff $a^2 + b^2 = c^2 + d^2$***

(i) verify that (A, R) is an equivalence relation.

(ii) describe geometrically what the equivalence classes are for this reason.

SOLUTION. (i) We have $(a, b)R(c, d) \Rightarrow a^2 + b^2 = c^2 + d^2$

$\Rightarrow c^2 + d^2 = a^2 + b^2 \Rightarrow (c, d)R(a, b)$ ...(1)

$\Rightarrow$ R is symmetric.

Now, $(a, b)R(c, d)$ and $(c, d)R(x, y) \Rightarrow a^2 + b^2 = c^2 + d^2$

and $c^2 + d^2 = x^2 + y^2$

$\Rightarrow a^2 + b^2 = x^2 + y^2 \Rightarrow (a, b)R(x, y)$ ...(2)

$\Rightarrow$ R is transitive.

Again $(a, b)R(a, b) \Leftrightarrow a^2 + b^2 = a^2 + b^2$ ...(3)

$\Rightarrow$ R is reflexive.

Hence, from (1), (2) and (3), we conclude that R is an equivalence relation.

(ii) For any point (a, b), the sum $a^2 + b^2$ is the square of the distance from the origin. The equivalence classes are, therefore, the set of points in the place which have the same distance from the origin. Hence, the equivalence classes are concentric circles centered on the origin.

EXAMPLE 11. ***Let R be the binary relation defined as $R = \{(a, b) \in R^2 : a - b \le 3\}$. Determine whether R is reflexive, symmetric, anti symmetric and transitive.***

SOLUTION. We have $(a, b) \in R^2 : a - b \le 3$.

$\Rightarrow$ $(a, a) \in R^2 : a - a \le 3$ *i.e.*, $0 \le 3$, which is true. So, R is reflexive.

In a similar way, we can easily show that R is neither symmetric, anti symmetric nor transitive.

1.13.8 RELATIONS OTHER THAN EQUIVALENCE

Let R be a given relation on the set X. Then R is

(i) non-reflexive if $\exists x$, such that $(x, x) \notin R$.

(ii) anti-reflexive or reflexive if $I_x \cap R = \phi$ (where I_x is the identity relation on X or $\forall n \in X : (x,x) \notin R$

(iii) non-symmetrical if for some $(x, y) \in R$, we have $(y, x) \notin R$

(iv) asymmetric if $R \cap R^{-1} = I$, *i.e.*, $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$

(v) anti-symmetric if $R \cap R^{-1} = \phi$, *i.e.*, $(x, y) \in R \Rightarrow (y, x) \notin R$

(vi) non-transitive if $R \circ R \notin R$

(vii) anti-transitive if $(R \circ R) \cap R = \phi$

(viii) A reflexive and symmetric, but not transitive relation is called a tolerance relation.

(ix) A non-symmetric transitive relation is called an ordered relation.

(x) A reflexive, anti-symmetric and transitive relation is called partial-ordered relation.

EXERCISE 1.6

1. If R is the relation 'is less than' from $A = \{1, 2, 3, 4, 5\}$ to $B = \{1, 4, 5\}$, find the set of ordered pairs corresponding to R. Also find R^{-1}.
2. A relation R defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows : $(x, y) \in R \Rightarrow x$ divides y. Write R as a set of ordered pairs and determine the domain and range of R. Also find R^{-1}.
3. Find the domain and range of $A = \{1, 2, 3, 4, 5, 6\}$ when the relation are defined as
   (i) xR_1y if and only if $x - y > 0$
   (ii) xR_2y if and only if $x + y < 0$
4. Two sets A and B are given by $A = \{1, 2, 8, 9\}$ and $B = \{2, 3, 4, 6, 7\}$ and if R is the relation form A to B given by $\{(1,2), (1,3), (2,4), (2,6)\}$, then which of the following statement is true?
   (i) Domain $(R) \neq$ Range (R^{-1}) and Range (R) = Domain (R^{-1})
   (ii) Domain (R) = Domain (R^{-1}) and Range (R) = Range (R^{-1})
   (iii) Domain (R) = Range (R^{-1}) and Range (R) = Domain (R^{-1})
   (iv) Domain (R) = Range (R)
5. If R is a relation on a set A, then which of the following statement is not true?
   (i) If R is reflexive then R^{-1} is reflexive.
   (ii) If R is symmetric then R^{-1} is symmetric.
   (iii) If R is transitive, then R^{-1} is transitive.
   (iv) None of these
6. Find the domain and range of the following relations:
   (i) $R = \{(x + 1, x + 5)\} : x \in \{0, 1, 2, 3, 4, 5\}$
   (ii) $R = \{(x, x^3) : x$ is a prime number, less than $10\}$
   (iii) $R = \{(a, b) : a \in \boldsymbol{N}, a < 5, b = 4\}$
   (iv) $R = \{(a, b) : b = |a - l|, a \in \boldsymbol{Z}$, and $|a| \leq 3\}$
7. Let R_1 be the relation defined on the set of reals $\boldsymbol{R}$ such as $(a, b) \in R_1$ if and only if $1 + ab > 0$ for all $a, b \in \boldsymbol{R}$. Show that R_1 is reflexive, symmetric but not transitive.
8. Let R be relation on $\boldsymbol{N} \times \boldsymbol{N}$, defined by $(a, b) R (c, d)$ if and only if and $(b + c) = bc(a + d)$. Show that R is an equivalence relation.
9. Show that the relation 'congruence modulo m' on the set of integers is an equivalence relation.
10. Let R_1 be a relation on the set of reals defined by $R_1 = \{(a, b) \in R \times R : a^2 + b^2 = 1\}$
    Show that R_1 is not an equivalence relation on R.
11. In a set L of all straight lines in a plane, discuss which of the following two relations are equivalence relations L.
    (i) $R_1 = \{(x, y): x, y \in L$ and x is parallel to $y\}$
    (ii) $R_2 = \{(x, y): x, y \in L$ and x is perpendicular to $y\}$.
12. Show that the relation
    $R = \{(a, b) : a - b =$ even integr $\forall\, a, b \in \boldsymbol{Z}\}$, *i.e.*, $aRb \Leftrightarrow a - b =$ even integer, is an equivalence relation.
13. Show that the relation R in $\boldsymbol{N}$, the set of natural numbers, defined by xRy if $x^2 - 4xy + 3y^2 = 0$, $(x, y \in \boldsymbol{N})$ is reflexive, not symmetric and not transitive.
14. For the given relation R on a set S, determine which are equivalence relations:
    (i) S is the set of all rational numbers, aRb if and only if $a = b$
    (ii) S is the set of all real numbers iff
    (a) $|a| = |b|$ (b) $a \geq b$
    (iii) S is the set of all triangles in a plane, aRb iff a is congruent to b.
    (iv) S is the set of all triangles in a plane, aRb iff a and b have equal perimeters.
15. An integer m is said to be related to another integer n if m is a multiple of n. Show that this relation is reflexive and transitive but not symmetric.
16. Let R be a relation defined on the set of natural number $\boldsymbol{N}$ as $R = \{(x, y): x, y \in \boldsymbol{N}, 2x + y = 41\}$. Find the domain and range of R.

17. Let O be the origin. Define a relation between two points P and Q in a plane if $PO = OQ$. Show that the relation is an equivalence relation.

18. Given the relation $R = \{(1, 2), (2, 3)\}$ on the set of natural number $\boldsymbol{N}$, add a minimum of ordered pairs so that the enlarged relation is symmetric, transitive and reflexive.

19. Let $\boldsymbol{N}$ denote the set of all natural numbers and R be the relation on $\boldsymbol{N} \times \boldsymbol{N}$ defined by $(a, b) R (c, d) \Leftrightarrow ad\,(b + c) = bc\,(a + d)$. Show that R is an equivalence relation.

20. Show that the relation, which is symmetric and transitive, is not necessarily reflexive.

ANSWERS

1. $aRb = \{(1, 4), (1, 5), (2, 4), (3,4), (2, 5), (3, 5), (4, 5)\}$, $R^{-1} = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\}$

2. Domain $(R) = \{2, 3, 5\}$, Range $(R) = \{3, 6, 10\}$, $R^{-1} = \{(6, 2), (10, 2), (3, 3), (6, 3), (10, 5)\}$

3. (i) $\{2, 3, 4, 5, 6\}$, $\{1, 2, 3, 4, 5\}$, (ii) ϕ, ϕ **4.** (iii) **5.** (iv) **6.** (i) Domain $(R) = \{1, 2, 3, 4, 5, 6\}$, Range $(R) = \{5, 6, 7, 8, 9, 10\}$ (ii) Domain $(R) = \{2, 3, 5, 7\}$, Range $(R) = \{8, 27, 125, 243\}$ (iii) Domain $(R) = \{1,2,3,4\}$, Range $(R) = \{4\}$ (iv) Domain $(R) = \{0, -1, -2, -3, 1, 2, 3\}$, Range $(R) = \{1, 2, 3, 4, 0, 1, 2\}$ **11.** R_1 = Equivalence relation, R_2 = Not equivalence **14.** (i), (ii) **16.** Domain $(R) = \{1, 2, \ldots, 19, 20\}$, Range $(R) = \{39, 37, 35, \ldots, 5, 3, 1\}$ **18.** $\{(1, 2), (2,1), (2, 3), (3, 2), (1, 3), (3, 1), (1,1), (2, 2), (3, 3), (4, 4), \ldots\}$

1.14 FUNCTIONS

Definition: *Let A and B be two sets, then the rule or corresponding, which associates each element of A to a unique element to B, is called a function from set A to set B.*

If a general element of set A is denoted by x, and of set B is denoted by y, then we say that y is a function of x if, for every $x \in A$, one and only one value of $y \in B$ can be determined.

Symbolically: If f is a function from a set A to a set B, then we write $f : A \to B$, read as f is a function from A to B or f maps A to B.

1.14.1 RANGE AND DOMAIN OF A FUNCTION

Let an element $y \in B$ be corresponded by an element $x \in A$, then y is called the image of x and is denoted by $f(x)$. Here, x is defined as the pre-image of y.

The set A is called the domain and the set B is called the co-domain of the function f.

The set of all f-images of the element of A, is called image set or the range of f and is denoted by

$$f(A) \quad \text{or} \quad \{f(x) : x \in A\}$$

Evidently, $f(A) \subseteq B$.

Thus, a mapping $f : A \to B$ is the set of ordered pairs $\{(a, b) : a \in A, b \in B\}$, so that no two ordered pairs have the same finite element.

$$f = \{(a, b) : a \in A, b \in B, b = f(x) \ \forall\ a \in A\}$$

For example: Let $A = \{-2, -1, 0, 1, 2\}$ and B is the set of natural numbers for every $x \in A$, $f(x) \in B$ and $f(x) = x^2$.

Here, A is the domain and B is the co-domain.

$f(a)$ is the value of the function $f(x)$, when x takes the value a, *i.e.*, when x is replaced by a.

The elements of the co-domain which is equal to $f(x)$ form the range.

When $x = -2$, $f(-2) = (-2)^2 = 4$

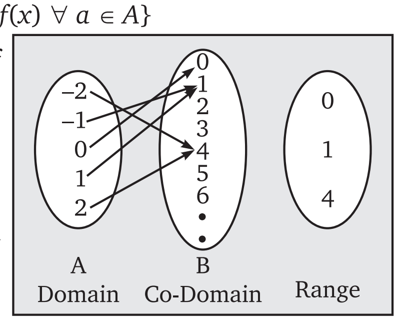

Fig. (15)

When $x = -1, f(-1) = 1$
When $x = 0, f(0) = 0$
When $x = 1, f(1) = 1$
When $x = 2, f(2) = 4.$
Which can be illustrated in the figure (15).

REMARKS

- If $f : A \to B$ then a single element in A cannot have more than one image in B. However, two or more elements in A may have the same images in B.
- Every element in A must have its image in B, but every element in B may not have it pre-image in A.
- To each element x in A, there exists a unique element y in B such that $y = f(x)$.
- The unique element y of B is called the value of f at x (the image of f under x), and written as $y = f(x)$.
- The range of f consist of those elements in B which appear as the image of at least one element in A.
- Range of a function is the image of its domain.
- Range is a subset of co-domain.

1.15 TYPES OF FUNCTIONS

1.15.1 ONE-ONE FUNCTION

A function f from A to B, *i.e.*, $f : A \to B$ is said to be one-one (or injective) iff distinct elements of A have distinct images.

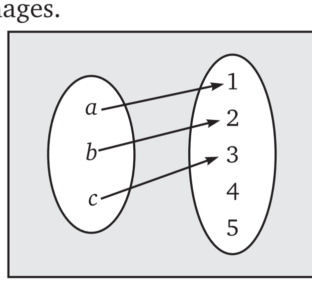

Fig. (16)

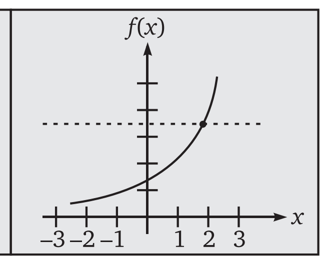

Fig. (17)

Symbolically: f is one-one if for $x_1, x_2 \in A$, we have

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2) \ \forall \ x_1, x_2 \in A$$

or $\quad f(x_1) = f(x_2) \ \Rightarrow \quad x_1 = x_2 \ \forall \ x_1, x_2 \in A$

It is also called **Univalent function.**

Graphically, a function is one-one if and only if no line parallel to x-axis meets the graph of the function in more than one point.

1.15.2 MANY-ONE FUNCTION

A function $f : A \to B$ is called many-one, if at least one element of co-domain B has two or more than two pre-images in domain A.

Symbolically: f is many-one if for $x_1, x_2 \in A$, we have $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$

This can be illustrated in the 18 and 19 figure.

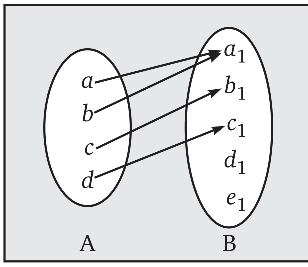

Fig. (18)

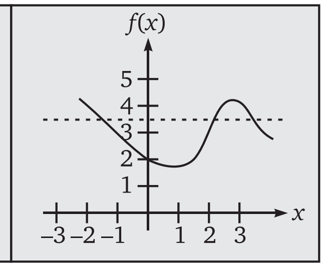

Fig. (19)

Graphically, a function is many-one if and only if a line parallel to x-axis meets the graph of the function in more than one point.

REMARK

➡ One-many function does not exist.

1.15.3 ONTO FUNCTION

A function $f : A \to B$ is called an onto function, if there is no element of B which is not an image of some element of A, *i.e.,* every element of B appears as the image of at least one element of A. This is illustrated in Figure 20.

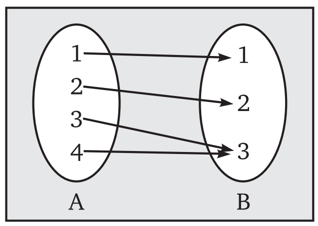

Fig. (20): Onto Function

REMARKS

➡ In an onto function, Range = Co-domain

➡ Onto function is also called surjective.

1.15.4 INTO FUNCTION

A function $f : A \to B$ is called an into function, *i.e.*, if there is at least one element of set B which has no pre-image in the set A. This is illustrated in Figure 21.

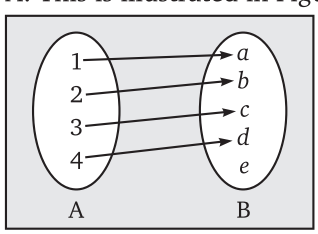

Fig. (21): Into Function

REMARK

➡ In an into function, Range $\subset$ Co-domain.

1.15.5 ONE-ONE INTO FUNCTION

A function $f : A \to B$ is called a one-one into function, if it is both one-one and into, *i.e.*, the different points in A are joined to different points in B and there are some points in B which are not joined to any point in A. This is illustrated in Figure 22.

Symbolically : One-one into function is defined as

(i) Range $\subset$ Co-domain.

(ii) $f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2.$

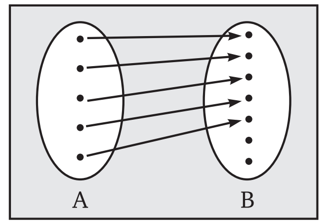

Fig. (22): One-One Into Function

1.15.6 ONE-ONE ONTO FUNCTION

A function $f : A \rightarrow B$ is both one-one and onto, *i.e.*, the different points in A are joined to different points in B and no point in B is left vacant. This is illustrated in Figure 23.

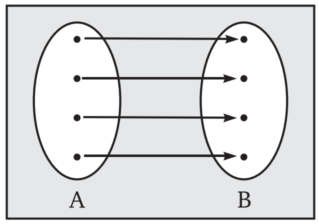

Fig. (23): One-one Onto Function

REMARKS

- One-one onto mapping is also known as bijective or one-to-one.
- For a one-one onto function, Range = Co-domain, and $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

1.15.7 MANY-ONE INTO FUNCTION

A function $f : A \rightarrow B$ which is both many-one and into function is called a many-one into function, i.e., two or more points in A are joined to some points in B and there are some point in B which are not joined to any point in A. Therefore, for many-one into function.

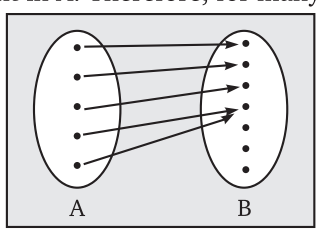

Fig. (24): Many-One Into Function

(i) Range $\subset$ Co-domain.

(ii) $x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) = f(x_2)$

1.15.8 MANY-ONE ONTO FUNCTION

If function $f : A \rightarrow B$ is both many-one and onto function is called a many one onto function, *i.e.*, in B one point is joined to at least one point in A and two or more points in A are joined to some points in B. Therefore, for many-one onto function.

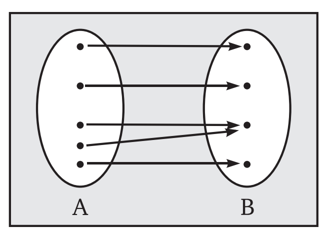

Fig. (25): Many-One Onto Function

(i) Range = Co-domain.

(ii) $x_1 \neq x_2 \quad \Rightarrow f(x_1) = f(x_2)$

WORKING PROCEDURE

1. **For checking the Injectivity (One-One) of the function**

   Let x and y be two arbitrary elements in the domain of f.

STEP 1. Take $f(x) = f(y)$

STEP 2. If we get $x = y$, after solving $f(x) = f(y)$. Then, $f : A \to B$ is one-one.

2. **For checking the surjectivity (Onto) of a function**

STEP 1. Take an arbitrary element y in the co-domain.

STEP 2. Put $f(x) = f(y)$

STEP 3. Solve $f(x) = y$ for x and obtain x in terms of y.

STEP 4. Get the equation of the form $x = g(y)$

STEP 5. If $x = g(y)$ belongs to domain f, for all values of y, then f is onto.

Solved Examples

EXAMPLE 1. ***Let $f : R \to R$ be a function defined by***

$$f(x) = \begin{cases} 3x - 1 & \text{when} \quad x > 3 \\ x^2 - 1 & \text{when} \quad -2 \le x \le 3 \\ 2x + 3 & \text{when} \quad x < -2 \end{cases}$$

Find (i) $f(2)$, (ii) $f(4)$, (iii) $f(-1)$, (iv) $f(-3)$

SOLUTION. (i) $f(2) = (2)^2 - 1 = 4 - 1 = 3$

(ii) $f(4) = 3(4) - 1 = 12 - 1 = 11$

(iii) $f(-1) = (-1)^2 - 1 = 1 - 1 = 0$

(iv) $f(-3) = 2(-3) + 3 = -6 + 3 = -3$

EXAMPLE 2. ***For $y = +\sqrt{x}$, say whether it is a function or not. If it is a function, find its domain and range.***

SOLUTION. Here we have $y = +\sqrt{x}$ ...(1)

Since y is real if $x \ge 0$ and is unique and finite for each $x \ge 0$.

Therefore, (1) is a function with domain $[0, \infty[$.

Again from (1), $y \ge 0 \ \forall \ x \ge 0$

Hence, range $= [0, \infty[$

EXAMPLE 3. ***Find the domain of $f(x) = \dfrac{x^3 - x^2 + 4x + 2}{3x + 11}$.***

SOLUTION. Since f is defined for all real values of x except when $3x+11 = 0$

i.e., when, $x = -\dfrac{11}{3}$

Hence, domain of $f = \mathbf{R} - \left\{-\dfrac{11}{3}\right\}$

EXAMPLE 4. ***Let $f : N - \{1\} \to N$ be defined by $f(n)$ = the highest prime factor of n. Show that f is neither one-one nor onto. Also, find the range f.***

SOLUTION. Since we have

$f(6)$ = the highest prime factor of $6 = 3$

$f(9)$ = the highest prime factor of $9 = 3$

$f(12)$ = the highest prime factor of 12 = 3

Therefore, f is a many-one function.

Clearly, image of any $n \in \boldsymbol{N} - \{1\}$ is the largest prime number that divides n. So the range of f consists of prime number only. Consequently, range of $f \neq \boldsymbol{N}$ (Co-domain)

$\Rightarrow$ f is not onto function.

Hence, f is neither one-one nor onto. The range of f is the set of all prime numbers.

EXAMPLE 5. ***Let $A = \{1, 2\}$. Find all one-to-one function from A to A.***

SOLUTION. Let $f : A \to A$ be a one-one function.

Then, for $f(1)$, there are two choices, *i.e.,* 1 or 2.

Let us first suppose $f(1) = 1$.

As $f : A \to A$ is one-one, $f(2) = 2$

Therefore, we have $f(1) = 1, f(2) = 2$

Now, let $f(1) = 2$

Since, $f : A \to A$ is one-one, therefore $f(2) = 1$.

Therefore, we have $f(1) = 2$ and $f(2) = 1$.

Hence, we have two one-one function say f and g form A and A given by $f(1) = 1$, $f(2) = 2$ and $f(2) = 1$ and $f(1) = 2$.

EXAMPLE 6. ***Let $\{x \in \mathbf{R} : -1 \leq x \leq 1\} = B$. Show that $f : A \to B$ given by $f(x) = x\,|x|$ is one-one and onto.***

SOLUTION. Let x, y be any two elements in A, then

$$x \neq y \Rightarrow x|x| \neq y|y| \Rightarrow f(x) \neq f(y).$$

Therefore, f is one-one.

Since, range of $f = f(A) = B$ so $f : A \to B$ is onto mapping. Hence f is one-one and onto.

EXAMPLE 7. ***Find the domain and range of the function $f(x) = -\sqrt{-5 - 6x - x^2}$.***

SOLUTION. Given that, $f(x) = -\sqrt{-5-6x-x^2}$

For f to be real, $-5 - 6x - x^2 \geq 0$ $\quad\Rightarrow\quad x^2 + 6x + 5 \leq 0$

$\Rightarrow\quad x^2 + 6x \leq -5 \quad\Rightarrow\quad x^2 + 6x + 9 \leq -5 + 9$

$\Rightarrow\quad (x + 3)^2 \leq 4 \quad\Rightarrow\quad |x + 3|^2 \leq 4$

$\Rightarrow\quad |x + 3| \leq 2 \quad\Rightarrow\quad -2 \leq x + 3 \leq 2$

$\Rightarrow\quad -2 - 3 \leq x \leq 2 - 3 \quad\Rightarrow\quad -5 \leq x \leq -1$

Therefore, domain of $f(x) = [-5, -1]$

To find the range of $f(x)$, put $y = f(x)$

Therefore, $f(x) = -\sqrt{-5-6x-x^2}$, $y \leq 0$

$\Rightarrow\quad y^2 = -5 - 6x - x^2 \quad\Rightarrow\quad x^2 + 6x + (y^2 + 5) = 0$

For real x, discriminant ≥ 0, $\quad$ *i.e.,* $(6)^2 - 4 \times 1 \times (y^2 + 5) \geq 0$

$\Rightarrow\quad 36 - 4y^2 - 20 \geq 0 \quad\Rightarrow\quad -4y^2 \geq -16$

$\Rightarrow\quad y^2 \leq 4 \quad\Rightarrow\quad |y|^2 \leq 4$

$\Rightarrow\quad |y| \leq 2 \quad$ *i.e.,* $\quad -2 \leq y \leq 2$

But $y \leq 0$ therefore, $-2 \leq y \leq 0$.

Hence, Range of $f = [-2, 0]$

EXAMPLE 8. ***For a finite set A, if f : A → A is a one-one function, show that f is onto.***

SOLUTION. Let $A = \{a_1, a_2, ..., a_n\}$ be a finite set.

Since $f : A \to A$ is one-one function, therefore $f(a_1), f(a_2), ..., f(a_n)$ are distinct elements of the set A, but A has only n elements. Therefore,

$$A = \{f(a_1), f(a_2), ..., f(a_n)\}$$

$\Rightarrow$ Co-domain = Range

Hence, every element in A (co-domain) has its pre-image in the domain A.

$\Rightarrow$ $f : A \to A$ is onto.

REMARK

- For a finite set A, if $f : A \to A$ is onto function, then f is one-one.

EXAMPLE 9. ***If f : R → R be a function defined by f(x) = 4x³ – 7, show that the function f is bijective.***

SOLUTION. Given that $f(x) = 4x^3 - 7; x \in \mathbf{R}$

***f* is one-one :** Let $x_1, x_2 \in \mathbf{R}$

Now, $f(x_1) = f(x_2)$

$\Rightarrow \quad 4x_1^3 - 7 = 4x_2^3 - 7 \qquad \Rightarrow \quad 4x_1^3 = 4x_2^3$

$\Rightarrow \quad x_1^3 = x_2^3 \qquad \Rightarrow \quad x_1^3 - x_2^3 = 0$

$\Rightarrow \quad (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$\Rightarrow \quad (x_1 - x_2)\left[\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3x_2^2}{4}\right] = 0 \qquad \left\{\because \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3x_2^2}{4} \neq 0\right\}$

$\Rightarrow \quad (x_1 - x_2) = 0 \qquad \Rightarrow x_1 = x_2$

Therefore, f is one-one.

***f* is onto :** Let $c \in \mathbf{R}$

$f(x) = c \quad \Rightarrow 4x^3 - 7 = c \qquad \Rightarrow x = \left(\frac{c+7}{4}\right)^{1/3}$

Now, $\left(\frac{c+7}{4}\right)^{1/3} \in \mathbf{R}$ and $f\left\{\left(\frac{c+7}{4}\right)^{1/3}\right\} = 4\left[\left(\frac{c+7}{4}\right)^{1/3}\right]^3 - 7 = c + 7 - 7 = c$

Which implies that c is the image of $\left(\frac{c+7}{4}\right)^{1/3}$

Therefore, f is onto. Hence, f is bijective function.

EXAMPLE 10. ***Let A and B be two sets. Prove that f : A × B → B×A difined by f(a,b)=(b,a) is one-one and onto.***

SOLUTION. ***f* is one-one :** Let (a_1, b_1) and $(a_2, b_2) \in A\times B$ such that

$$f(a_1, b_1) = f(a_2, b_2)$$

$\Rightarrow \quad (b_1, a_1) = (b_2, a_2)$

$\Rightarrow \quad b_1 = b_2$ and $a_1 = a_2$

Therefore, $\quad (a_1, b_1) = (a_2, b_2)$

Thus, $\quad f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow \quad (a_1, b_1) = (a_2, b_2) \; \forall \; (a_1, b_1), (a_2, b_2) \in A\times B$

$\Rightarrow \quad f$ is one-one.

***f* is onto :** Let $(b, a) \in B\times A$ such that $b \in B$ and $a \in A$.

$\Rightarrow \quad (a, b) \in A\times B$

Therefore, for all $(b, a) \in B\times A$, there exist $(a, b) \in A\times B$ such that $f(a, b) = (b, a)$
$\Rightarrow$ f is onto. Hence f is one-one and onto.

EXERCISE 1.7

1. Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to \boldsymbol{Z}$ given by $f(x) = x^2 - 2x - 3$. Find :
   (i) the range of f,
   (ii) pre-image of 6, –3 and 5.
2. Find the domain and range of the following function
   $$f(x) = \sqrt{(x-1)(3-x)}$$
3. Find the range of the following function
   $$f(x) = \frac{1}{(2x-3)(x+1)}$$
4. Find the domain and range of the following functions :
   (i) $f(x) = \dfrac{x^2-1}{x-1}$ (ii) $y = -|x|$
   (iii) $f(x) = \dfrac{|x-1|}{x-1}$ (iv) $y = \sqrt{x-3}$
5. If $A = \{-1, 0, 2, 5, 6, 11\}$,
   $B = \{-2, -1, 0, 18, 25, 108\}$
   and $f(x) = x^2 - x - 2$, find $f(A)$.
6. Let A be the set of two positive integers. Let $f : A \to \boldsymbol{Z}^+$, set of positive integers be defined by $f(n) = p$, where p is the highest prime factor of n. If range of $f = \{3\}$, find A.
7. Find the domain for which the function $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.
8. Let $f_1 : \boldsymbol{R} \to \boldsymbol{R}$ and $f_2 : \boldsymbol{C} \to \boldsymbol{C}$ be two functions defined as $f_1(x) = x^3$ and $f_2(x) = x^3$. Show that they are not equal.
9. Let $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$. Which of the following relations from A to B not a funcion?
   (i) $R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$
   (ii) $R_2 = \{(p, 1), (q, 1), (r, 1), (s, 1)\}$
   (iii) $R_3 = \{(p, 1), (q, 2), (r, 2), (s, 3)\}$
   (iv) $R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$
10. Write the following relations as sets of ordered pairs and find which of them are functions :
    (i) $\{(x, y) : y = 3x, x\in(1, 2, 3), y\in(3, 6, 9, 12)\}$
    (ii) $\{(x, y) : y > x + 1, x = 1, 2$ and $y = 2, 4, 6\}$
    (iii) $\{(x, y) : x + y = 3$ $x, y\in(0, 1, 2, 3)\}$
11. Express the following functions as sets of ordered pairs, and find their range :
    (i) $f_1 : A \to \boldsymbol{R} : f_1(x) = x^2 + 1$ where $A = \{-1, 0, 2, 4\}$
    (ii) $f_2 : A \to \boldsymbol{N} : f_2(x) = 2x$ where $A = \{x : x \in \boldsymbol{N}, x \leq 10\}$
12. Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a function such that $f(x) = 2^x$. Determine :
    (i) range of f
    (ii) $\{x : f(x) = 1\}$
    (iii) whether $f(x + y) = f(x)\cdot f(y)$ holds
13. Let $f : \boldsymbol{R}^+ \to \boldsymbol{R}$, be a function such that $f(x) = \log x$. Determine :
    (i) the image set of domain of f
    (ii) $\{x : f(x) = -2\}$
    (iii) whether $f(xy) = f(x) + f(y)$ holds
14. Give an example of a map which is :
    (i) one-to-one but not onto
    (ii) not one to one, but onto
    (iii) neither one-to-one nor onto

ANSWERS

1. (i) $f(A) = \{-4, -3, 0, 5\}$, (ii) ϕ, $\{1, 2\}$, -2 **2.** Domain = [1, 3], Range = [–1, 1] **3.** $\left]-\infty\ \frac{-8}{25}\right] \cup [0, \infty[$
4. (i) $\boldsymbol{R} - \{1\}$, $\boldsymbol{R} - \{2\}$, (ii) $\boldsymbol{R} : \boldsymbol{R} - \boldsymbol{R}^+$, (iii) $\boldsymbol{R} - \{1\}$, $\{-1, 1\}$, (iv) $[3, \infty[$, $[0, \infty]$
5. $f(A) = \{1, -2, 18, 28, 108\}$ **6.** $A = \{3, 6\}$ or $(3, 9)$ or $[3, 12]$ etc. **7.** $(-2, 1/2)$ **9.** (iii)
10. (i) $\{(1, 3), (2, 6), (3, 9)\}$, function, (ii) $\{(1, 4), (1, 6), (3, 4), (3, 6)\}$, not function
(iii) $\{(0, 3), (1, 2), (2, 1), (3, 0)\}$, function
11. (i) $f_1 = \{x, f(x) : x \in A\} = \{(-1, 2), (0, 1), (2, 5), (4, 17)\}$
(ii) $f_2 = \{(x, g(x)) : x \in A\} = \{(1,2),(2,4), (3, 6), ..., (10, 20)\}$
12. (i) Range of $f = \boldsymbol{R}^+$, the set of positive real numbers, (ii) $(x : f(x) = 1) = \{0\}$,

(iii) $f(x+y) = f(x).f(y)$ holds for all $x, y \in \boldsymbol{R}$

14. (i) $n \to n^2 : \boldsymbol{N} \to \boldsymbol{N}$ (ii) $n \to |n| : \boldsymbol{Z} \to \boldsymbol{N} \cup \{0\}$ (iii) $n \to |n|^2 : \boldsymbol{Z} \to \boldsymbol{N} \cup \{0\}$

1.16 BOUNDEDNESS OF A SUBSET OF REAL NUMBERS

1.16.1 UPPER BOUND OF A SUBSET OF R

A subset S of $\boldsymbol{R}$ is said to be bounded above, if there exists a real number u such that $s \leq u$ $\forall s \in S$. The real number u is said to be upper bound of S.

If there exists no such upper bound, then the set is said to be unbounded above.

ILLUSTRTIONS

- The set of natural number $\boldsymbol{N} = \{1, 2, 3, ...\}$ is not bounded above or unbounded above.
- The set of positive integers $\boldsymbol{Z}^+$ is not bounded above.
- The set $S = \{1, 2, 3, 4\}$ is bounded above by 4.
- The set $\left\{\frac{1}{n} : n \in \boldsymbol{N}\right\}$ is bounded above by 1.
- The set of negative integers is bounded above by 0.

1.16.2 LOWER BOUND OF A SUBSET OF R

A subset S of $\mathbf{R}$ is said to be bounded below if there exists a real number l such that $s \geq l$ $\forall\ s \in S$. The real number l is said to be the lower bound of S and if there exists no such lower bound, then the set is said to be unbounded below.

ILLUSTRATIONS

- The set of natural numbers, $\boldsymbol{N}$ is bounded below by 1.
- The set $\left\{\frac{1}{n} : n \in \boldsymbol{N}\right\}$ is bounded below by 0.
- The set $S = \{1, 2, 3, 4\}$ is bounded below by 1.
- The set of positive real numbers is bounded below.

1.16.3 BOUNDED SET

A subset S of $\boldsymbol{R}$ is said to be a bounded if it is bounded below as well as bounded above *i.e.*, if there exist two real numbers l and u such that

$$l \leq s \leq u, \ \forall\ s \in S.$$

Equivalently, if there exists an interval I (= $[l, u]$) such that $S \subseteq I$

ILLUSTRATIONS

- Every finite set is bounded.
- The set $\left\{\frac{1}{n} : n \in \boldsymbol{N}\right\}$ is bounded.

1.16.4 UNBOUNDED SET

A subset S of $\boldsymbol{R}$, which is not bounded is called an unbounded set.

ILLUSTRATIONS

- The sets $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ are unbounded sets.
- Set of all prime numbers is an unbounded set.

REMARKS

- If a set is bounded above, then it has infinitely many upper bounds in as much as every number greater than an upper bound is also an upper bound.
- If a set is bounded below, then it has infinitely many lower bounds in as much as every number smaller than a lower bound is also a lower bound.
- It is not necessary that lower bounds and upper bounds of a set S are the members of S.
- The null set ϕ is bounded but it is neither possesses lower bound nor upper bound.

1.17 SUPREMUM AND INFIMUM OF A SET

1.17.1 LEAST UPPER BOUND (OR SUPREMUM)

A real number u is said to be a least upper bound of a set S if

(i) u is an upper bound of S

and (ii) if u' is an another upper bound of S then $u \le u'$.

i.e., no real number less than u can be an upper bound of S.

1.17.2 GREATEST LOWER BOUND (OR INFIMUM)

A real number l is called a greatest lower bound of a set S if

(i) l is a lower bound of S

and (ii) if l' is another lower bound of S, then $l' \le l$

i.e., no real number greater then l can be a lower bound of S.

For Example :

If $S = \left\{\frac{1}{n} : n \in \boldsymbol{N}\right\}$ then l.u.b. = 1 and g.l.b. is 0.

REMARKS

- If a real number u is the supremum of a subset S of real numbers then, for every $\varepsilon > 0$, there exists a real number $x \in S$ such that $u - \varepsilon < x < u$.
- If a real number is the infimum of a subset S of real numbers then, for every $\varepsilon > 0$, there exists a real number $x \in S$ such that $l \le x < l + \varepsilon$.
- Supremum is defined only for the bounded above sets and infimum for the subset, which are bounded below.
- The supremum and infimum of a set may or may not belong to the set.
- If supremum of a set belongs to the set, then supremun is the largest element of the set.
- If infimum of a set belongs to the set, then infimum is the smallest element of the set.
- Supremum and infinium of a bounded subset of $\boldsymbol{R}$, are unique.
- In case of singleton set $S = [a]$, $a \in \boldsymbol{R}$, supremum and infimum coincide.
- If u and l are the supremum and infimum of a non-empty subset S of $\boldsymbol{R}$, then $l \le u$.

THEOREM I. ***The supremum of a set $S \subset R$, if exists, is unique.***

PROOF. Let S be a non-empty subset of $\boldsymbol{R}$.

Let if possible, s_1 and s_2 be two supremum of S.

To show $s_1 = s_2$.

Since we assume that s_1 and s_2 are the supremums of S.

$\Rightarrow$ s_1 and s_2 are the upper bounds of S.

Let us first suppose s_1 is a supremum and s_2 is an upper bound of S, then

$$s_1 \le s_2. \quad \ldots(1)$$

Now, if s_2 is the supremum and s_1 is the upper bound of S, then

$$s_2 \leq s_1. \quad ...(2)$$

From (1) and (2), $s_1 = s_2$.

Hence, supremum of a set, if exists is unique.

THEOREM 2. ***The infimum of a set $S \subset R$, if exists, is unique.***

PROOF. Proof is similar as theorem 1 and left to the reader.

THEOREM 3. ***If S is a non-empty subset of R, then a real number s is the supremum of S if and only if***

(i) $x \leq s \ \forall x \in S$

and (ii) for each positive real number ε, ***there exists a real number*** $x \in S$ ***such that*** $x > s - \varepsilon$.

PROOF. **Necessary Condition** (only if part).

Let us first suppose s be the supremum of the set S.

Let s be the supremum of $S \Rightarrow s$ is an upper bound of S.

By definition $x \leq s \ \forall x \in S$.

Let $\varepsilon > 0$ be any real number. Then obviously $s - \varepsilon < s$

$\Rightarrow$ $(s - \varepsilon)$ is not an upper bound of S. ($\because$ s is l.u.b. of S.)

Hence, there must exist some $x \in S$ such that $x > s - \varepsilon$.

Sufficient part (If part)

Let us suppose condition (i) and (ii) holds.

Then, to show $s = \sup S$

By condition (i), we have s is an upper bound of S. To show s is the supremum of S, for this, it is enough to show that no real number less than s can be an upper bound of S.

Let s_1 be any real number less than s

$$\Rightarrow s - s_1 > 0.$$

Let us take $\varepsilon = s - s_1 \quad \Rightarrow \quad \varepsilon > 0.$

Then by condition (ii), these exists $x \in S$ such that $x > s - \varepsilon$

$\Rightarrow \quad x > s-(s - s') \quad \Rightarrow \quad x > s', x \in S$

$\Rightarrow$ s' is not an upper bound of S.

Hence, we can say that s is an upper bound of S and no real number less than s is an upper bound of S.

$\Rightarrow$ s is the supremum of S.

THEOREM 4. ***Let S be a non-empty subset of R, then a real number t is the infimum of S if and only if***

(i) $x \geq t$ ***for all*** $x \in S$ ***and***

(ii) For each real number $\varepsilon > 0$, ***there exists a real number*** $x \in S$ ***such that*** $x < t + \varepsilon$.

PROOF. Proof is similar as theorem 3.

Solved Examples

EXAMPLE 1. ***Show that***

(i) The set R^+ ***of positive real numbers, is bounded below and unbounded above.***

(ii) The set R^- of negative real numbers, is bounded above and unbounded below.

SOLUTION. (i) Since every member of $\boldsymbol{R}^- \cup \{0\}$ is lower bound of $\boldsymbol{R}^+$, therefore $\boldsymbol{R}^+$ is bounded below.

To prove $\boldsymbol{R}^+$ is unbounded above.

Let if possible, suppose u is an upper bound of R^+ we have $u \geq 1$ for $1 \in \boldsymbol{R}^+$. Since $2 \in \boldsymbol{R}^+$, $2 > 0$ and so $u \geq 1$, $2 > 0$ gives $u + 2 > 1 + 0$ *i.e.*, $u + 1 > 0$. Thus $(u + 1) \in \boldsymbol{R}^+$ and $(u + 1) > u$ which is a contradiction, that u is an upper bound of $\boldsymbol{R}^+$.

Hence, $\boldsymbol{R}^+$ is unbounded above.

(ii) Proof follows in a similar manner.

EXAMPLE 2. ***Show that the set of real numbers, R is an unbounded set.***

SOLUTION. From example 1, we conclude that the set $\boldsymbol{R}^+$ is unbounded above and $\boldsymbol{R}^-$ is always unbounded below.

Also $\qquad \boldsymbol{R} = \boldsymbol{R}^- \cup \{0\} \cup \boldsymbol{R}^+$

$\Rightarrow$ $\boldsymbol{R}$ is not bounded.

EXAMPLE 3. ***Show that the null set ϕ is neither bounded below nor above, nor unbounded.***

SOLUTION. Since, there is no member in ϕ, we cannot check whether a given real number can be a bound for ϕ or not. Thus, bounds for ϕ do not exists. On the other hand we can as well say that every real number is a lower or upper bound for there is no member in ϕ which does not satisfy the required property of bounds.

EXAMPLE 4. ***Show that every non-empty finite subset of R is bounded.***

SOLUTION. Let S be a non-empty finite subset of $\boldsymbol{R}$.

$\Rightarrow S$ contains a finite number of elements. Then by the properties of the ordered relation in $\boldsymbol{R}$, out of these elements, one element $s \in S$, shall be the smallest element of S and another element $b \in S$, shall be the greatest element of S

$\Rightarrow \qquad a \leq x \leq b \ \forall x \in S.$

Hence, S is always bounded.

EXAMPLE 5. ***Find the supremum and infimum of the set $S = \{x \in Z : x^2 \leq 25\}$.***

SOLUTION. Since

$$S = \{x \in \boldsymbol{Z} : x^2 \leq 25\}$$
$$= \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}.$$

Since S is a finite subset of $\boldsymbol{R}$, the smallest member of S is -5, which is a lower bound of S, and hence infimum of S is -5. Similarly 5 is the supremum of S.

EXAMPLE 6. ***Find the supremum and infimum, if they exist, of the following sets***

(i) $\left\{\dfrac{1}{n} : n \in N\right\}$ ***(ii)*** $\left\{x \in Q : x = \dfrac{n}{n+1},\ n \in N\right\}$

(iii) $\left\{1 + \dfrac{(-1)^n}{n} : n \in N\right\}$ ***(iv)*** $\left\{\pi + \dfrac{1}{2}, \pi + \dfrac{1}{4}, \pi + \dfrac{1}{8}, \ldots\right\}$

SOLUTION. (i) Here we have

$$S = \left\{\frac{1}{n} : n \in \boldsymbol{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}.$$

The set S is bounded above by 1, also any member less than 1 is not an upper bound of S, therefore sup S = 1.
Also, 0 is a lower bound of S, because $x \geq 0$, $\forall\, x \in S$. Let l be any arbitrary positive small number, then there exists $n \in \boldsymbol{N}$ such that $\frac{1}{n} < l$, which shows that l is not an upper bound of S. Thus 0 is a lower bound of S and no other positive real number is a lower bound of S. Therefore, infimum of $S = 0 \notin S$.

(ii) Let $$S = \left\{\frac{n}{n+1} : n \in \boldsymbol{N}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$$

Then, the set S is bounded below by $\frac{1}{2}$ and any number greater than $\frac{1}{2}$ can not be a lower bound of S, therefore infimum of $S = \frac{1}{2}$.

Also, $\left(\frac{n}{n+1}\right) < 1$, $\forall\, n \in \boldsymbol{N}$, therefore 1 is an upper bound of S, and any number less than 1 not be a upper bound of S.

Therefore supremum of S = 1.

(iii) Let $$S = \left\{1 + \frac{(-1)^n}{n} : n \in \boldsymbol{N}\right\} = \left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, \ldots\right\}$$
$$= \left\{\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \ldots, \frac{2n-2}{2n-1}\right\} \cup \left\{\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \ldots, \frac{2n+1}{2n}\right\}.$$

Here, the proper fraction $\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$ are increasing and tending to 1, and the improper fractions begin with $\frac{3}{2}$ are decreasing and tending to 1.

Therefore, infimum of S = 0 and supremum of $S = \frac{3}{2}$.

(iv) Let $$S = \left\{\pi + \frac{1}{2}, \pi + \frac{1}{4}, \pi + \frac{1}{8}, \ldots\right\}.$$

Here, we have $x \leq \pi + \frac{1}{2}$ $\forall\, x \in S$

$\Rightarrow$ $\pi + \frac{1}{2}$ is an upper bound for S.

Since, $\pi + \frac{1}{2} \in S$, therefore no real number less than $\pi + \frac{1}{2}$ can be upper bound for S. Thus, $\pi + \frac{1}{2}$ is the least upper bound. Therefore, supremum of $S = \pi + \frac{1}{2}$.

Similarly, we can show that π is the infimum of S.

1.18 SEQUENCES

Let $\boldsymbol{N}$ be the set of natural numbers and S be any set of real numbers. A function, whose domain is the set of natural numbers and range is a subset of S, is called a sequence in S.

Symbolically. If we define a function $f : \boldsymbol{N} \to S$, then f is a sequence. We shall denote a sequence in a number of ways as follows :

(i) Usually, a sequence is denoted by its images. For a sequence f, the image corresponding to $n \in \boldsymbol{N}$ is denoted by f_n or $<f(n)>$ and $f(n)$ is called the n^{th} term of the given sequence. For example $<1, 4, 9, \ldots>$ is the sequence whose n^{th} term is n^2.

(ii) Using in order, the first few element of a sequence, here the rule for writing down different elements becomes clear. For example, $<1, 2, 3, \ldots>$ is the sequence whose n^{th} term is n.

(iii) Defining a sequence by a recurrence formula, *i.e.*, by a rule which express the n^{th} term by $(n-1)^{\text{th}}$ term. For example, let $a_1 = 1, a_{n+1} = 2a_n \quad \forall n \geq 1$.

These above relations define a sequence whose n^{th} term is 2^{n-1}.

Remarks

- A sequence is represented as $<s_n>$ or $\{s_n\}$, when s_n is the n^{th} term of the sequence. The set of all distinct terms of a sequence is called the range set of that sequence.
- A sequence, whose range is a subset of real numbers $\boldsymbol{R}$ is called a real sequence or a sequence of real numbers.

1.18.1 CONSTANT SEQUENCES

A sequence $< s_n >$ defined by $s_n = a, \ \forall n \in \boldsymbol{N}$, is called a constant sequence.

1.18.2 EQUALITY OF SEQUENCES

Two sequence $< s_n >$ and $< t_n >$ are said to be equal, if $s_n = t_n \quad \forall n \in \boldsymbol{N}$.

1.18.3 OPERATION ON SEQUENCES

Since, the sequence are real valued functions, therefore, the sum, difference, product etc. of two sequence are defined as follows :

(i) If $< s_n >$ and $< t_n >$ be any two sequences, then the sequence, whose nth terms are $s_n + t_n$, $s_n - t_n$ and $s_n . t_n$ are respectively known as the sum, difference and product of the sequence $< s_n >$ and $< t_n >$ and are denoted by $< s_n + t_n >, < s_n - t_n >$ and $< s_n . t_n >$ respectively.

(ii) If $s_n \neq 0 \quad \forall n \in \boldsymbol{N}$, then the sequence, whose nth term is $\frac{1}{s_n}$, is called reciprocal of the sequence $< s_n >$ and is denoted by $< \frac{1}{s_n} >$.

(iii) The sequence, whose nth term $s_n / t_n (t_n \neq 0 \quad \forall n \in \boldsymbol{N})$ is known as the quotient of the sequence $< s_n >$ by the sequence $< t_n >$ and is denoted by $< \frac{s_n}{t_n} >$.

(iv) The sequence, whose nth term is ksn, where $k \in \boldsymbol{R}$ is known as the scalar multiple of the sequence $< s_n >$ by k and is denoted by $< ks_n >$.

1.18.4 BOUNDED SEQUENCE

(i) Bounded below sequence. A sequence $< s_n >$ is said to be bounded below if there exists a real number l such that $s_n \geq l \quad \forall n \in \boldsymbol{N}$.

The number l is known as the lower bound of the sequence $< s_n >$.

(ii) Bounded above sequence. A sequence $< s_n >$ is said to be bounded above if there exists a real number u such that $s_n \le u \ \ \forall n \in \boldsymbol{N}$.

The number u is called upper bound of the sequence $< s_n >$.

(iii) Bounded sequence. A sequence $< s_n >$ is said to be bounded if it is bounded above as well as bounded below.

(iv) Unbounded sequence. A sequence $< s_n >$ is said to be unbounded if it is not bounded.

(v) Least upper bound. If a sequence $< s_n >$ is bounded above, then there exists a number u_1 such that

$$s_n \le u_1 \quad \forall n \in \boldsymbol{N}. \qquad \ldots(1)$$

This number u_1 is called an upper bound of the sequence $< s_n >$. If $u_1 < u_2$. Then from (1), we find that

$$s_n < u_2 \quad \forall n \in \boldsymbol{N}$$

which implies, u_2 is also an upper bound of the sequence $< s_n >$. Hence, we can say that any number greater than u_1 is an upper bound of $< s_n >$.

Hence, a sequence has an infinite number of upper bounds, if it is bounded above. Let u be the least of all the upper bound of the sequence $< s_n >$. Then u is defined as the least upper bound (l.u.b) or supremum of the sequence $< s_n >$.

(vi) Greatest lower bound. If a sequence $< s_n >$ is bounded below, then there exists a number $l_1 \in \boldsymbol{R}$ such that

$$l_1 \le s_n \quad \forall n \in \boldsymbol{N} \qquad \ldots(2)$$

This number l_1 is known as the lower bound of $< s_n >$. If $l_2 < l_1$, then from (2)

$$l_2 \le s_n \quad \forall n \in \boldsymbol{N}$$

which implies, l_2 is also a lower bound of the sequence $< s_n >$. Hence, we can say that any number less than l_1 is a lower bound of $< s_n >$.

Hence, a sequence has infinite number of lower bounds, if it is bounded below. Let l be the greatest of all the lower bounds of the sequence $< s_n >$. Then l is known as greatest lower bound $(g.l.b)$ or infimum of the sequence $< s_n >$.

☛ ILLUSTRATIONS

- ✦ The sequence $<n^2>$ is bounded below by 1 but not bounded above.
- ✦ The sequence $< \frac{n}{n+1} >$ is bounded as $\frac{1}{2} \le \frac{n}{n+1} < 1 \quad \forall n \in \boldsymbol{N}$.
- ✦ The sequence $< \frac{1}{n} >$ is bounded since $\left|\frac{1}{n}\right| \le 1 \quad \forall n \in \boldsymbol{N}$.
- ✦ The sequence $<2^n>$ is bounded below and has smallest term as 2. Every member of $]-\infty \ 2]$ is a lower bound of the sequence and the sequence is not bounded above.

1.18.5 LIMIT POINT OF THE SEQUENCE

A real number l is called a limit point of a sequence $< s_n >$ if every nbd of l contains infinite number of terms of the sequence.

Thus, $l \in \boldsymbol{R}$ is a limit point of the sequence $< s_n >$ if for given $\varepsilon > 0, s_n \in \ ]l - \varepsilon, l + \varepsilon]$ for infinitely many points.

Here it must be noted that

(i) limit point of a sequence need not be a member of the sequence.

(ii) A limit point of a sequence may or may not be a limit point of the range of the sequence but the limit point of the range of a sequence is always a limit point of the sequence.
(iii) In case of real numbers, limit points of a sequence may also be called accumulation, cluster or condensation points.

☛ **ILLUSTRATIONS**

✦ The sequence $< \frac{1}{n} >$ has one limit point, *i.e.*, 0.

✦ The sequence $< (-1)^n >$ has two limit points 1 and –1.

✦ The sequence $< n >$ has no limit point.

✦ The sequence $< 1 + \frac{(-1)^n}{n} >$ has one limit point, *i.e.*, 1.

1.18.6 SUFFICIENT CONDITIONS FOR NUMBER l TO BE OR NOT BE A LIMIT POINT OF THE SEQUENCE $< s_n >$

(i) If for every $\varepsilon > 0, \exists m \in \boldsymbol{N}$ such that $s_n \in \,]l-\varepsilon, l+\varepsilon[\; \forall n \geq m$ or equivalently $|s_n - l| < \varepsilon \;\; \forall n \geq m$, then l is the limit point of the sequence $< s_n >$.
(ii) If for any $\varepsilon = 0, s_n \in \,]\,l-\varepsilon, l+\varepsilon[$ for only a finite number of values of n, then l is not a limit point of the sequence $< s_n >$. Such a condition is also necessary for a number l not to be limit point of the sequence $< s_n >$.

1.18.7 BOLZANO-WEIRSTRASS THEOREM FOR SEQUENCE

[KANPUR–2000]

STATEMENT. ***Every bounded sequence has at least one limit point.***

PROOF. Let $S = \{s_n : n \in \boldsymbol{N}\}$ be the range set of the bounded sequence $< s_n >$.

Then, S is a bounded set. Now, there may be two cases :

(i) Let S be a finite set. Then $s_n = p$ for infinitely many indices n. Here $p \in \boldsymbol{R}$. Obviously p is a limit point of $< s_n >$.
(ii) Let S be an infinite set. Since, S is bounded, then by Bolzano-Weirstrass theorem for set of real numbers, S has a limit point say p. Therefore, every nbd of p contains infinity many distinct points of S, *i.e.*, infinitely many term of $< s_n >$ and hence p is a limit point of the sequence $< s_n >$.

1.18.8 LIMIT SUPERIOR AND LIMIT INFERIOR

The greatest limit point of a bounded sequence is called the upper limit or limit superior and is denoted by $\overline{\lim}\, s_n$ and the smallest limit point of a bounded sequence is called the lower limit or limit inferior and is denoted by $\underline{\lim}\, s_n$

- By definition, it is obvious that $\underline{\lim}\, s_n \leq \overline{\lim}\, s_n$.
- A bounded sequence $< s_n >$ for which the upper limit and lower limit coincide with real number l is said to converge to l.

1.18.9 LIMIT OF A SEQUENCE

A sequence $< s_n >$ is said to have a limit l if for a given $\in > 0$ $\exists$ a positive integer m such that

$$|s_n - l| < \in \quad \forall n \geq m$$

1.18.10 CONVERGENT SEQUENCE

A sequence $< s_n >$ is said to converge to a number l, if for a given $\in > 0$ there exists a positive integer m such that

$$|s_n - l| < \in \quad \forall n \geq m$$

REMARK

- A sequence $<s_n>$ is said to be convergent iff it is bounded and has exactly one limit point.

1.18.11 DIVERGENT SEQUENCE

A sequence, which is not convergent, is known as divergent sequence.

1.18.12 OSCILLATORY SEQUENCE

A sequence $< s_n >$ is said to be an oscillatory sequence if it is neither convergent nor divergent.

An oscillatory sequence is said to be oscillate finitely or infinitely according as it is bounded or unbounded.

In other words, we can say

(i) a bounded sequence, which is not convergent is said to be oscillate finitely.

(ii) an unbounded sequence, which does not diverge, is said to be oscillate infinitely.

(iii) a bounded sequence, which does not converge and has at least two limit points is said to be oscillate finitely.

ILLUSTRATIONS

- The sequence $\left\langle 1+(-1)^n \right\rangle$ oscillate finitely.
- The sequence $\left\langle (-1)^n \right\rangle$ oscillate finitely.
- The sequence $<(-1)^n\left(1+\dfrac{1}{n}\right)>$ oscillate finitely.
- The sequence $<n(-1)^n>$ oscillate infinitely.

EXERCISE 1.8

1. Find the supremum and infimum, if exist of the following sets.

   (i) $\left\{\dfrac{1}{5n} : n \in \boldsymbol{Z}, n \neq 0\right\}$

   (ii) $\left\{x : x = 1+\dfrac{1}{n} : n \in \boldsymbol{N}\right\}$

   (iii) $\{x \in \boldsymbol{R} : x = 2^n : n \in \boldsymbol{N}\}$

   (iv) $\{3, 8, 14, 20\}$

   (v) $\left\{-\dfrac{1}{n} : n \in \boldsymbol{N}\right\}$

   (vi) $\left\{m+\dfrac{1}{n} : m, n \in \boldsymbol{N}\right\}$

   (vii) $\left\{x \in \boldsymbol{Q} : x = \dfrac{(-1)^n}{n} : n \in \boldsymbol{N}\right\}$

   (viii) $S = \left\{\left(1-\dfrac{1}{n}\right)\sin\dfrac{n\pi}{2} : n \in \boldsymbol{N}\right\}$

   (ix) $\left\{x = (-1)^n\left(\dfrac{1}{4}-\dfrac{4}{n}\right) : n \in \boldsymbol{N}\right\}$

2. Find the supremum and infimum of the set S= $\left\{\dfrac{2n+1}{3n+3} : n \in \boldsymbol{N}\right\}$.

3. Prove that every subset of a bounded above (below or both) set is bounded above (below or both).

4. If A and B are subsets of $\boldsymbol{R}$, then prove that the set $A + B = \{x + y : x \in A, y \in B\}$ is also bounded and

   $\inf (A + B) = \inf (A) + \inf (B)$.

5. If $A \neq \phi$ is bounded below and $-A$ denotes the set of all $-x$ for which $x \in A$, then prove that $-A \neq \phi$, that $-A$ is bounded above and that $-\sup(-A) = \inf (A)$.

6. If $A \neq \phi, B \neq \phi$ and $x \leq y \ \forall\ x \in A$ and $y \in B$, then show that

   (i) $\sup A \leq y \ \forall\ y \in B$ (ii) $\sup (A) \leq \inf (B)$.

7. If $A \subseteq B$ and B is bounded, then show that

   $\sup B \geq \sup A \geq \inf A \geq \inf B$.

8. If A and B are two bounded subset of $\boldsymbol{R}$, then $A \cup B$ and $A \cap B$ ($\neq \phi$) are also bounded and
   (i) $\sup(A \cup B) = \max(\sup A, \sup B)$
   (ii) $\inf(A \cup B) = \min(\inf A, \inf B)$
   (iii) $\sup(A \cap B) = \min(\sup A, \sup B)$
   (iv) $\inf(A \cap B) = \max(\inf A, \inf B)$.
9. Give an example of a set in which supremum is equal to infimum.
10. Show that the set $[x : x \in \boldsymbol{Q}, x>0$ and $x^2 < 3]$ does not have any supremum in $\boldsymbol{Q}$.
11. For a real number λ and a subset A of $\boldsymbol{R}$, let λA be the set defined by $\lambda A=\{\lambda x : x\in A\}$. Prove that if A is bounded, then λA is also bounded and
$$\inf(\lambda A) = \begin{cases} \lambda \inf(A) \text{ if } \lambda \geq 0 \\ \lambda \sup(A) \text{ if } \lambda \leq 0. \end{cases}$$
12. Give an example of a set which is
   (i) bounded above but not below.
   (ii) bounded below but not above.
   (iii) neither bounded above nor bounded below.
   (iv) both bounded above and below.
13. Find the supremum and infinium of the sets
   (i) $S = \{x \in \boldsymbol{Z} : x^2 \leq 16\}$
   (ii) $S = \left\{2 + \frac{1}{n} : n \in \boldsymbol{N}\right\}$
14. Check the boundedness of the following sets:
   (i) $\{-1, -2, -3, ...\}$
   (ii) $\{1, 2, 3, 4, 5, ...\}$
   (iii) $\{2, 2^2, 2^3, ..., 2^n, ...\}$
   (iv) $\left\{1, \frac{1}{4}, \left(\frac{1}{4}\right)^2, \left(\frac{1}{4}\right)^3, ..., \left(\frac{1}{4}\right)^n, ...\right\}$
   (v) $\left\{x : x = (-1)^n \frac{1}{n} : n \in \boldsymbol{N}\right\}$
   (vi) $\{x : x = (-2)^n : n \in \boldsymbol{N}\}$

Answers

1. (i) inf $= -\frac{1}{5}$, sup $= \frac{1}{5}$ (ii) sup = 2, inf =1 (iii) inf = 2, sup, does not exist
   (iv) inf = 3, sup = 20 (v) sup = 0, inf = – 1 (vi) inf = 1, sup, does not exist
   (vii) inf = –1, sup = 1/2 (viii) inf = –1, sup = 1 (ix) sup = 15/4, inf = –7/4
2. sup $= \frac{2}{3}$, inf $= \frac{3}{5}$
9. Singlton set **12.** (i) [–1, –2, –3, ...] (ii) $\boldsymbol{N}$ (iii) $\boldsymbol{Z}$, $\boldsymbol{Q}$ or $\boldsymbol{R}$ (iv) Any finite set.
14. (i) Bounded above but not below (ii) Bounded below but not above
   (iii) Bounded below but not above (iv) Bounded below and above, both
   (v) Bounded below and above both (vi) Neither bounded below nor above.

Review Questions And Archive

1. Define union, intersection, difference and symmetric difference of two sets.
2. Define the power set of a set
3. How many element does the power set of a set S with n elements have?
4. Define what it mean for a function from the set of positive integers to the set of positive integer to be one to one.
5. Define the inverse of a function.
6. Define he floor and ceiling functions from the set of real numbers to the set of integers.
7. Let $f(n)$ be the function from the set of integers to the set of integers such that $f(n) = n^2 + 1$. What are the domain, co-domain and range of this function.
8. Give an example of a function from the set of positive integers to the set of positive integers that is :
   (a) both one-one and onto.
   (b) one-one but not onto.

(c) neither one-one nor onto.
(d) not one-one but is onto.

9. When the empty set the power set of a set?

10. (a) Define what is means for two sets to be equal.
(b) Describe the ways to show that two sets are equal.

11. Let A and B be sets in a finite universal set U. List the following in order of increasing size:
(a) $|A|$, $|A \cup B|$, $|A \cap B|$, $|U|$, $|\phi|$
(b) $|A - B|$, $|A \oplus B|$, $|A| + |B|$, $|A \cup B|$, $|\phi|$

12. Research where the concept of a function first arose and describe how this concept was first used.

MULTIPLE CHOICE QUESTIONS (CHOOSE THE MOST APPROPRIATE ONE)

1. Let R_1 and R_2 be two equivalence relation on a set. Consider the following assertion :
(i) $R_1 \cup R_2$ is an equivalence relation.
(ii) $R_1 \cap R_2$ is an equivalence relation.
Which of the following is correct?
(a) Both assertions are true.
(b) Assertion (i) is true but assertion (ii) is not true.
(c) Assertion (ii) is true but assertion (i) is not true.
(d) Neither (i) or (ii) is true.

2. The 'subset' relation on a set of set is :
(a) a partial ordering
(b) an equivalence relation
(c) transitive and symmetric only
(d) transitive and anti-symmetric only

3. Let R be a symmetric and transitive relation on a set A, then :
(a) R is reflexive and hence an equivalence relation.
(b) R is reflexive and hence a partial order.
(c) R is not reflexive and hence is not an equivalence relation .
(d) None of the above

4. The number of equivalence relations of the set {1, 2, 3, 4} is :
(a) 4 (b) 15
(c) 16 (d) 24

5. Suppose A is a finite set with n elements. The number of elements in the large equivalence relation of A is :
(a) 1 (b) n
(c) $n + 1$ (d) n^2

6. The binary relation $S = \phi$ on the set $A = \{1, 2, 3\}$ is :
(a) neither reflexive nor symmetric
(b) symmetric and reflexive
(c) transitive and reflexive
(d) transitive and symmetric

7. Let $f(x) = x^2 + x$ and $g(x) = x + 1$ then fog is :
(a) $x^2 + 3x + 2$ (b) $x^2 + x + 1$
(c) $(x+1)^2 + (x+1)$ (d) None of these

8. Let A and B be sets with cardinalities m and n respectively. The number of one-to-one mapping from A to B where $m < n$ is :
(a) m^n (b) ${}^nP_m$
(c) ${}^mC_n$ (d) ${}^nC_m$

9. The number of functions from m element set to n element set is :
(a) $m + n$ (b) m^n
(c) n^m (d) $m * n$

10. _________ is an unordered collection of elements where an element can occur as a member more than once :
(a) Multiset (b) Ordered set
(c) Set (d) None of these

11. The number of substrings of all lengths that can be formed from a character string of length n = ________
(a) n (b) n^2
(c) $\frac{n(n-1)}{1}$ (d) $\frac{n(n+1)}{2}$

12. In a room containing 28 females, there are 18 females who speak English, 15 females speak French and 22 speak German. 9 females speak both English and French, 11 Females speak both French and German whereas 13 speak both German and English. How many females speak all the three langulages?

(a) 9 (b) 8
(c) 7 (d) 6

13. Consider the following statements :

***S_1* :** There exist infinite set A, B and C such that $A \cap (B \cap C)$ is finite.

***S_2* :** There exist two irrational numbers x and y such that $(x + y)$ is rational.

Which of the following is True about S_1 and S_2?

(a) Only S_1 is correct.
(b) only S_2 is correct.
(c) Both S_1 and S_2 are correct.
(d) None of the S_1 and S_2 is correct.

14. The power set 2^S of the set $S = \{3,\{1,4\},5\}$ is:

(a) $\{5, 3, 1, 4, \{1, 3, 5\},\{1, 4, 5\},\{3,4\}, \phi\}$
(b) $\{5, 3, \{1, 4\}, 5\}$
(c) $\{5, \{3\}, \{3, \{1,4\}\}, \{3, 5\}, \phi\}$
(d) None of the above

15. Let A be a finite set of size n, the number of elements in the power set of $A \times A$ is :

(a) 2^n (b) 2^{n^2}
(c) $(2^n)^2$ (d) None of these

16. Let S be an infinite set and $S_1, S_2, S_3, \ldots S_n$ be the sets such that $S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n = S$. Then :

(a) at least one of the set S_i is a finite set.
(b) not more than one of the set S_i can be finite.
(c) at least one of the sets S_i is an infinite set.
(d) None of the above

17. The Number of elements in the power set $P(S)$ of the set $S = \{\{\phi\}, 1, \{2, 3\}\}$ is :

(a) 2 (b) 4
(c) 8 (d) None of these

18. Let A and B be sets and A' and B' denote the complements of the sets A and B. The set $(A - B) \cup (B - A) \cup (A \cap B)$ is equal to :

(a) $A \cup B$ (b) $A' \cup B'$
(c) $A \cap B$ (d) $A' \cap B'$

Answers

1. (c)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (d)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (c)
10. (a)	**11.** (d)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (a)

Learner's Diary

A Glimpse of Extra Facts

- $A \times B$ is an infinite set if either A or B or both are infinite sets.
- Every relation has an inverse relation.
- An equivalence relation ~ over a set induces a partition of the set. Conversly, a partition of a set defines an equivalence relation.
- A partition of a set into mutually exclusive disjoint subsets defines an equivalence relation.
- The relation in ***N*** defined by '$x < y$' is a partial order and is called natural order or usual order in ***N***.
- If $f : X \to Y$ and $g : A \to Y$ be two maps where $A < X$ such that $g(x) = f(x)\ \forall\ x \in A$. Then g is called restriction of f to A and is denoted by $f|A$ or f_A. Also, f is called extension of g.
- A linearly ordered set $(S, \leq)$ is said to be well ordered if every subset of A has a first element.
- Ordinal numbers corresponding to finite ordered sets are called finite ordinals.
- Ordinal numbers corresponding to infinite well ordered sets are called transfinite ordinals.
- Cartesian product of a non-empty set of family of non-empty sets is non-empty.
- Let $B = \{A_n : n \in \Delta\}$ be a family of pairwise disjoint non-empty sets. Then there exists a set A such that A_n contains exactly one element for each $n \in \Delta$.
- A family $\mathcal{F}$ is said to be of finite character if $A \in \mathcal{F} \Leftrightarrow B \in \mathcal{F}\ \forall\ B \subset A$ such that B is finite.