CHAPTER 1

Introduction

1.1 INTRODUCTION

The calculus (differential and integral) briefly called 'the Calculus and mathematical analysis' emerged in 17th century as a powerful tool for bringing about a rapid development of modern mathematical science. The basic notions of calculus are

(i) derivative, and

(ii) integral

These notions can be properly understood without an adequate knowledge of the concept of numbers, limits and functions. We shall therefore deal with the concept of numbers first, and shall thereafter take up the concept of functions, sequence and countability in this chapter.

1.2 CONCEPT OF SETS

The theory of sets is one of the most important tools of pure mathematics. Pure mathematics is the study of sets equipped with assigned structures, knows as mathematical systems. In this section, we shall study some fundamental concepts of set theory.

Definition. *'A set is a well defined collection of objects'.*

The objects of a set are called the elements or members of that set and their membership is defined by certain conditions.

The sets are usually denoted by the capital letters of English alphabets: Say $A, B, C, \ldots, X, Y, Z$.

For Example

(i) The collection of the letters a, b, c, d, . . .

(ii) The collection of all natural numbers denoted by $\boldsymbol{N}$.

(iii) The students of M.Sc., Mathematics in C.C.S. University, Meerut.

(iv) The collection of vowels in English alphabets. This set containing only five elements, namely a, e, i, o, u.

(v) The collection of all states in Indian union.

If S is a set, an object a in the collection S is called an element of S. This fact is expressed in symbol as $a \in S$ (read as a is in S or a belongs to S). If a is not in S, we write $a \notin S$. For example $4 \in \boldsymbol{R}$, the set of real numbers, but $\sqrt{-2} \notin \boldsymbol{R}$.

Here, Greek letter $\in$ denotes 'belongs to'. It is the abbreviation of the Greek word meaning 'is'.

☛ REMARKS

➠ By the term 'well defined' we mean that we are given a collection of objects, with certain definite property, so that we are able to determine whether a given objects belongs to our collection or not. Thus, every collection of objects is not a set.

- Set and aggregate both have the same meaning.
- The elements of a set must be distinguished from one another. The collection of sand particles does not form a set.
- The collection of rich persons of a city is not a set. However the collection of those persons of city whose wealth exceeds, a fixed amount, say rupees ten thousands, is a set.
- The order is not preserved in case of a set, whereas order is necessarily preserved in case of sequence. That is to say, each of the sets {1, 2, 3}, {3, 2, 1}, {1, 3, 2} denotes the same set.
- The repetition of an element does not change the nature of a set i.e. each of the sets {1,2,3}, {1, 2, 2, 3}, {1, 3, 3, 2} denotes the same sets.

1.2.1 EMPTY SET

A set containing no elements is called empty set and is denoted by the symbol ϕ.

For Example:

(i) $\phi = \{x : x$ is a negative integer whose square is $-1\}$

(ii) $\phi = \{x : x$ is a natural number lying between 2 and 3 $\}$

(iii) $\phi =$ {the set of such persons, who never die}

(iv) $\phi = \{x : x$ is a real number, $x^2 < 0\}$

(v) $\phi = \{x : x$ is an even prime number greater than five$\}$

(vi) $\phi =$ {the set of real numbers which are solution of equation $x^2 + 1 = 0$ }

(vii) $\phi = \{x : x$ is a straight line passing through three distinct points on a circle$\}$

REMARKS

- The empty set is also known as null set or void set.
- In Roster method, the empty set is denoted by { }.
- To describe the null set, we can use any property, which is not true for any element.
- It is wrong to use the expression 'an empty' or 'a null set' as there is one and only one empty set though, it may have many descriptions. We shall always call 'The empty or the null set'.
- A set consisting of at least one element is called a non-empty or non-void set.
- $\{\phi\}$ is not a null set.

1.2.2 SINGLETON SET

Set containing only one element is a Singleton set. The set $\{a\}$ is a singleton set.

REMARKS

- {0} is not a null set, since it contains 0 as its member. It is a Singleton set.
- A room containing only one man is not same thing as a man. In a similar way, the singleton set $\{a\}$ is not the same thing as the element a.

FACTS : TO THE POINT

We have the following statements about the real numbers:

- $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$
- $|x| \leq \varepsilon \Leftrightarrow -\varepsilon \leq x \leq \varepsilon$
- $|x| < \varepsilon \Leftrightarrow -\varepsilon < x < \varepsilon$
- $x = 0$ if and only if either $|x| \leq \varepsilon$ for each $\varepsilon > 0$ or else $|x| < \varepsilon$ for each $\varepsilon > 0$
- $|x + y| \leq |x| + |y|$
- $|x - y| \leq ||x| - |y||$
- $|xy| = |x| \cdot |y|$
- If $x_i \geq y_i$ for each i then $\sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} y_i$
- If $x_i \geq 0$ for each i and $\sum_{i=1}^{n} x_i = 0$ then $x_1 = x_2 = \ldots = x_n = 0$
- Following are equivalent: $x \geq y; x - y \geq 0; -y \geq -x; 0 \geq y - x$
- If $x \geq y$ and $z \geq 0$ then $xz \geq yz$
- $x^2 = 0$ if and only if $x = 0$
- If $x > 0$ then $x^{-1} > 0$ and if $x < 0$ then $x^{-1} < 0$
- $x > y > 0 \Leftrightarrow y^{-1} > x^{-1} > 0$
- $\max\{x, y\} \geq 0$ if and only if $x \geq 0$ or $y \geq 0$
- $\max\{x, y\} > 0 \Leftrightarrow x > 0$ or $y > 0$
- $\min\{x, y\} \geq 0 \Leftrightarrow x \geq 0$ or $y \geq 0$
- $\min\{x, y\} > 0 \Leftrightarrow x > 0$ or $y > 0$
- $x > 0 \Leftrightarrow$ there exists a positive integer n such that $n > x^{-1}$
- $x \geq 0 \Leftrightarrow x \geq -\frac{1}{n}$ for all positive integer n

1.2.3 FINITE SET

A set is said to be finite if it consists of only finite number of elements.

Here, the process of counting the different elements comes to an end.

For Example:

(i) Set of natural numbers less than 50.
(ii) Set of all persons in a city.
(iii) Set of English alphabets.
(iv) Set of all persons on the earth.

1.2.4 INFINITE SET

A set which is not finite *i.e.* it contains infinite number of elements. Here, process of counting the different elements never comes to an end.

For Example :

(i) Set of natural number $\boldsymbol{N} = \{1, 2, 3, ...\}$
(ii) Set of all points of a plane.
(iii) Set of all even integers.
(iv) Set of rational numbers lying between two integers.

1.2.5 EQUAL SETS

Two sets are said to be equal if they contain exactly the same elements. For Example :

$A = \{x : x \text{ is a letter in the word 'Area'}\}$ *i.e.* $A = \{a, r, e\}$

and $B = \{y : y \text{ is a letter in the word 'ear'}\}$ *i.e.* $B = \{a, r, e\}$

Here, A and B are equal sets.

1.2.6 CARDINAL NUMBER OF A SET

The number of distinct elements contained in a finite set A is called cardinal number of A and is denoted by $n(A)$.

1.2.7 EQUIVALENT SETS

Two finite sets are said to be equivalent if they have the same cardinal number.

☛ REMARKS

- Equivalent sets are not always equal but equal sets are always equivalent.
- The number of distinct elements in a finite set is also called the order of the set. If the order of a set is zero, the set is empty.
- If the order of a set is one, the set is singleton.
- The order of an infinite set is never defined.

1.3 SET OF NUMBERS

The number system plays a key role in Mathematics. The real number system $\boldsymbol{R}$ is one of the most important and beautiful mathematical system. There are different ways of introducing the real number system, but the most common way is to start with Peano's Axioms for the natural numbers. The axioms for natural numbers, discovered by the Italian Mathematician Peano are

(i) 1 is a natural number.
(ii) Each natural number n has a successor $(n + 1)$.
(iii) Two natural numbers are equal if their successors are equal.
(iv) Except 1, each natural number is a successor of a natural number.
(v) Any set of natural numbers which contains 1 and the successor of every natural number k whenever it contain k is the set N of natural number.

☛ **REMARK**

➠ Axiom (v) is commonly known as the axiom of induction or principle of finite induction.

The above axioms completely define the set of natural number.

1.3.1 NATURAL NUMBERS

The numbers 1, 2, 3,... are called natural numbers. We represent the set of natural numbers by $\boldsymbol{N}$ *i.e.* $\boldsymbol{N} = \{1, 2, 3, \ldots\}$

The Peano' s axioms can be used to extend the set $\boldsymbol{N}$ of natural numbers to another large system, known as the set of integers.

1.3.2 INTEGERS

The numbers ..., –3, –2, –1, 0, 1, 2, 3, ... are called integers. We represent the set of integers by $\boldsymbol{Z}$ *i.e.* $\boldsymbol{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Integers can be used to define the rational numbers.

1.3.3 RATIONAL NUMBERS

Any number of the form p/q, where $p, q \in Z$, $q \neq 0$ and p, q have no common factor (except ± 1) is called a rational number.

The set of rational numbers is denoted by $\boldsymbol{Q}$.

$\therefore \ \boldsymbol{Q} = \{\frac{p}{q}; p, q \in \boldsymbol{Z}, q \neq 0\}$

☛ **REMKARK**

➠ The set of rational numbers consist of integers and fractions.

Any number which is not rational, is called an Irrational number. For example, $\sqrt{2}$, $\sqrt{3}$, etc. It should be noted that every rational number can be expressed as a terminating or recurring decimal whereas every irrational number can be expressed as a non-terminating infinite decimal.

1.3.4 REAL NUMBERS

A number which is either rational or irrational is called a real number. The set of real numbers is denoted by $\boldsymbol{R}$.

1.4 CARTESIAN PRODUCT OF SETS

Here, we shall study the cartesian product of sets, concept of relation and concept of function.

1.4.1 ORDERED PAIR

An ordered pair is a pair of entries, whose components occur in a specific order. It is written by listing the two components in the specific order, separating them by a comma and enclosing the pair in parenthesis.

Symbolically. If A and B are two non- empty sets, then by ordered pair of elements, we must mean a pair (a,b) such that $a \in A, b \in B$ in the order.

☛ **REMARKS**

➠ It may be noted that (a, b) is not the same as $\{a, b\}$. The former denotes an ordered pair where later denotes a set.

➠ $(a,b) \neq (b,a)$ unless $a = b$.

➠ Ordered pair may have the same first and second components *i.e.*, two elements of an ordered pair need not be distinct.

1.4.2 CARTESIAN PRODUCT OF TWO SETS

The set of all ordered pair $\{(a,b): a \in A, b \in B\}$ is called the cartesian product of two sets A and B. It is denoted by $A \times B$.

Symbolically. $A \times B = \{(a,b): a \in A, b \in B\}$.

For Example. If $A = \{2,3\}$ and $B = \{4,5,6\}$.

Then $A \times B = \{(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$.

☛ **REMARKS**

- $A \times B = \phi \Leftrightarrow A = \phi$ or $B = \phi$.
- If A and B are finite sets, then $n(A \times B) = n(A).\ n(B)$.
- If either A or B is an infinite set, then $A \times B$ is an infinite set.

1.5 RELATION

Let us take two sets of natural numbers N_1 and N_2. We define R as a relation between them such that N_2 is a square of N_1 .Then

$1R1,\ \ 2R4,\ \ 3R9,\ \ 4R16....$

In terms of ordered pair, we can write

$R = \{(1,1),(2,4),(3,9),(4,16)...\}$

$= \{(x,y): x,y \in \boldsymbol{N} \text{ and } y = x^2\}$.

Then relation from set N_1 to N_2 is a subset of $N_1 \times N_2$ such that $(x,y) \in \mathrm{R}$ iff $y = x^2$.

Definition. *Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.*

Symbolically. R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

☛ **REMARKS**

- If R is a relation from A to B, then A is called the domain and B the range of R.
- If R is a relation from a non-empty set A to non-empty set B and if $(a,b) \in R$. Then we write $a\ R\ b$ and read as 'a is related to b by relation R'.
- Any subset $A \times A$ defines a relation in A, known as binary relation.

✍ **ILLUSTRATIONS**

- If $a,b \in \boldsymbol{N}$ and R is defined as "a is a divisor of b", then R is a relation on $\boldsymbol{N}$.
  The subset $\boldsymbol{N} \times \boldsymbol{N}$, which correspond to the relation R is $R = \{(n,nr): n \in \boldsymbol{N}, r \in \boldsymbol{N}\}$.
- If R is a relation from set $A = \{1,2,3\}$ to the set $B = \{-1,-2\}$ defined by $x + y = 0$ then $R = \{(1,-1),(2,-2)\}$.
  Hence, Domain of R is $\{1,2\}$ and range $= \{-1,-2\}$.
- If $A = (a,b,c,d,e\}$ and $B = \{f,g,h,i\}$ and let
  $R = \{(a,g),(a,i),(d,h),(e,f)\}$ be a relation from A to B.
  Then domain of $R = \{a,d,e\}$
  Range of $R = \{g,i,h,f\}$.
- If $a,b \in \mathrm{R}$ and S is "$|a-b|$ is a rational number", then S is a relation on R.

1.5.1 TOTAL NUMBER OF RELATIONS

Let A and B be two non-empty finite sets consisting of p and q elements respectively, then $A \times B$ consists of pq ordered pairs. Therefore, total number of subset of $A \times B$ and hence number of relations form A to B is 2^{pq}.

REMARKS

- For a non-empty set $A, \phi \subset A \times A$, therefore, it is a relation on A. This relation is called void or empty relation on A.
- The void relation ϕ and the universal relation $A \times B$, are called trivial relation from A to B.
- The void and universal relation on set A are respectively the smallest and the largest relation on A.

1.5.2 IDENTITY RELATION

Let A be a set. The identity relation on A is the relation $I_A = \{(x,x) : x \in A\}$.

For Example. If $A = \{a,b,c\}$, then the relation $I_A = \{(a,a),(b,b),(c,c)\}$ is the identity relation. Here, $R = \{(a,a),(b,b)\}$ is not an identity relation on A as $c \in A$ but as $(c,c) \notin R$.

1.5.3 INVERSE OF A RELATION

Let A, B be two non-empty sets and R be a relation from a set A to B and let (x,y) be element of the subset A of $A \times B$ corresponding to the relation R from A to B.

To the relation R from the set A to the set B, there corresponds a relation from the set B to the set A called the inverse of the relation R, denoted by R^{-1}, such that the subset $B \times A$ corresponding to the relation R^{-1} is equal to $\{(y,x) : (x,y) \in A\}$ *i.e.*, $yR^{-1}x \Leftrightarrow xRy$.

ILLUSTRATIONS

- Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ be two sets and let $R = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$ be a relation from A to B, then $R^{-1} = \{(1,a),(2,a),(1,b),(2,b)\}$.
- If $A = \{1,2,3\}, B = \{5,6,7\}$ and let $\{(1,5),(2,5),(2,7)\}$ be a relation from A to B, then $R^{-1} = \{(5,1),(5,2),(7,2)\}$, which is a relation from B to A.
- The inverse of the relation 'is less than' in R is 'is greater than'.

REMARKS

- Sometimes, the inverse of a relation coincides with the relation itself. For example, the inverse of the relation 'perpendicular to' in the set of straight lines coincides with itself.

1.5.4 CLASSIFICATION OF RELATIONS

(i) Reflexive relation. Let R be a relation on a set A, then "A relation R is said to be reflexive if $(x,x) \in R$, $\forall x \in A$" i.e., xRx, $\forall x \in A$

For Example :

i. In the set Z of integers, a relation R is defined by xRy iff $x - y$ is divisible by 4. Then, R is a reflexive relation as xRx, $\forall x \in Z$ as $x - x = 0$, which is divisible by 4.
ii. The universal relation on a non-empty set A is reflexive.
iii. The relation 'is less than' (*i.e.*,<) in the set of natural number is not reflexive because no number satisfies the relation 'is less than' to itself.
iv. The relation 'is a factor of ' in the set of rational number is reflexive, since every rational number is a factor of itself.
v. The relation 'is less than or equal to" (*i.e.*, $\leq$) is in set of natural numbers is reflexive because $n \leq n$ $\forall n \in N$.

(ii) Symmetric relation. A relation R on a set A is said to be symmetric if $(y,x) \in R$ whenever $(x,y) \in R$ $\forall x, y \in R$.

For Example :

(i) Let l_1, l_2 be two lines such that l_1 is perpendicular to l_2 *i.e.*, $l_2 \perp l_1$ then $l_1 \perp l_2 \Rightarrow l_2 \perp l_1$. Therefore, the relation $\perp$ is symmetric.

(ii) The identity and universal relation on a non-empty set are symmetric relation.
(iii) The relation 'less than' for natural number is not symmetric.

(iii) Transitive relation. A relation R on a set A is said to be transitive iff
$(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R \forall x,y,z \in A$.

For Example:
(i) Let a,b,c be three number such that a is a factor of b and b is a factor c, then obviously a is a factor of c. Therefore, 'is a factor of' is a transitive relation.
(ii) If l_1,l_2,l_3 are three lines such that $l_1 \nparallel l_2$ and $l_2 \nparallel l_3$ then obviously l_1 is parallel to l_3. Therefore, the relation is not transitive.

(iv) Anti-symmetric relation. A relation R on a non-empty set A is said to be anti-symmetric iff $(x,y) \in R$ and $(y,x) \in R \Leftrightarrow x = y \quad \forall x,y, \in \mathrm{R}$.

REMARKS
- The identity relation R on a set A is an anti-symmetric relation.
- If $(x,y) \in R$ and $(y,x) \in R$, then it may be that $x = y$.
- The universal relation on a set A, containing at least two elements is not anti-symmetric.

(v) Equivalence relation. A relation R on a set A is said to be an equivalence relation if it is
(a) reflexive
(b) symmetric
(c) transitive

For Example: In the set of integers, a relation R is defined by xRy if and only if $x-y$ is divisible by 4, then R is an equivalence relation, since
(a) For xRx, $x-x=0$ is divisible by 4. Therefore, it is reflexive.
(b) For xRy, let $x-y=4m$ so $y-x=4n$, which is also divisible by 4, therefore yRx and hence it is symmetric also.
(c) For xRy let $x-y=4m$; for yRz, let $y-z=4n$. By adding these two equation, we get $x-z=4(m+n)$, which is always divisible by . Therefore xRz and hence R is transitive.

1.5.5 COMPOSITION OF RELATION

Let R_1 and R_2 be two relation from sets A to B and B to C respectively. Then we can define a relation $R_1 \, o \, R_2$ from A to C, such that $(x,z) \in R_1 \, o \, R_2$ if and only if there exists $y \in B$ such that
$(x,y) \in R_1$ and $(y,z) \in R_2$.
This relation is called composition of R_1 and R_2.

1.6 FUNCTION

Let A and B be two sets, then a rule or correspondence, which associates each element of A to a unique element of B is called a function or mapping from set A to set B.

Symbolically. If f is a function from a set A to set B then we write $f: A \to B$ read as f is a function from A to B or f maps A to B.

1.6.1 RANGE AND DOMAIN OF A FUNCTION

Let an element $y \in B$ be corresponded by an element $x \in A$, then y is called the image of x and is denoted by $f(x)$. Hence, x is defined as the pre-image of y.

'The set A is called the domain and the set B is called the co-domain of the function f'.

The set of all f-image of the element A is called image set or the range of f and is denoted by $f(A)$ or $\{f(x): x \in A\}$.

Evidently, $f(A) \subseteq B$.

Thus a mapping $f : A \to B$ is the set of ordered pairs $\{(a,b) : a \in A, b \in B\}$.

so that no two ordered pairs have the same first element *i.e.*,

$$f = \{(a,b) : a \in A, b \in B, b = f(x) \quad \forall a \in A\}.$$

For Example. Let $A = \{-2,-1,0,1,2\}$ and B is the set of all integers, then for every $x \in A, f(x) \in B$ where $f(x) = x^2$. Here, A is the domain and B is the co-domain.

Also, $f(a)$ is the value of the function $f(x)$, when x takes the value a. The element of the co-domain which are equal to $f(x)$ for $x \in A$, form the range of f.

When $x = -2$, $f(x) = f(-2) = (-2)^2 = 4$

When $x = -1$, $f(x) = 1$

When $x = 0$, $f(x) = 0$

When $x = 1$, $f(x) = 1$

When $x = 2$, $f(x) = 4$

This can be illustrated in the following figures.

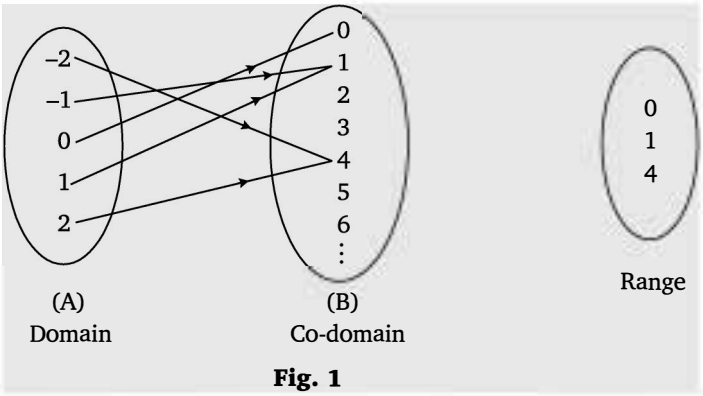


Fig. 1

☛ REMARKS

- If $f : A \to B$, then a single element in A, can not have more than one image in B. However, two or more element in A may have the same images in B.
- Every element in A must have its images in B, but every element in B may not have its pre-image in A.
- To each element x in A, there exists a unique element y in B such that $y = f(x)$.
- The range of f consists those element in B which appear as the image of at least one element in A. In other words, we can say, range of a function is the image of its domain.
- Range is a subset of co-domain.

1.6.2 TYPES OF FUNCTIONS

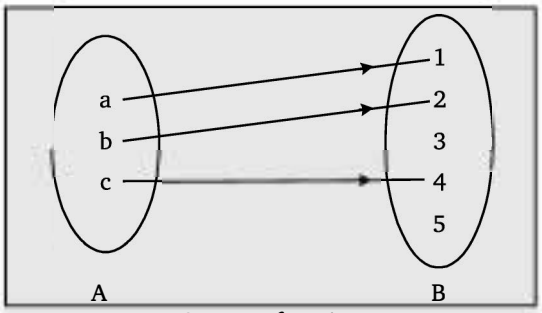
(a) One-one function. A function f from A to B *i.e.* $f : A \to B$ is said to be one-one (or injective) if and only if distinct elements of A have distinct images.

Symbolically. f is one-one if for $x_1, x_2 \in A$, we have

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \quad \forall x_1, x_2 \in A$$

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in A$.

It is also called injective function.



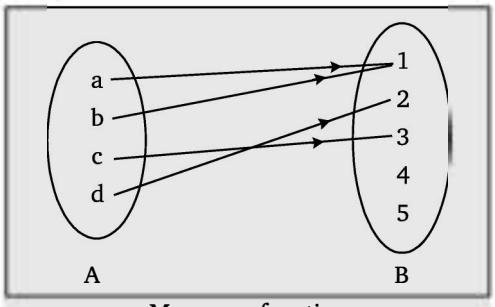
One-one function

Fig. 2

Graphically, a function is one-one if and only if no line parallel to x-axis meets the graph of the function in more than one point.

(b) Many-one function. A function $f : A \to B$ is called many-one, if at least one element of co-domain B has two or more than two pre-image in domain A.

Symbolically. f is many one, if for some $x_1, x_2 \in A$, we can have $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$. This can be illustrated in the following figure.



Many-one function

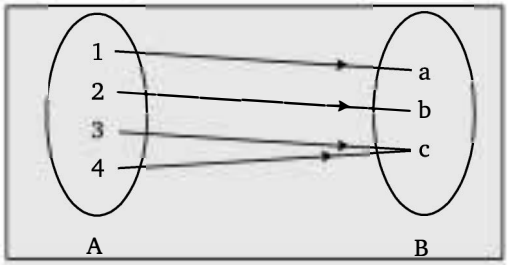
Fig. 3

Graphically, a function is many-one if and only if a line parallel to x-axis meets the graph of the function is more than one point.

☛ **REMARK**

➠ One-many function does not exist.

(c) Onto function. A function $f : A \to B$ is called an onto function, if there is no element of B, which is not an image of some element of A, *i.e.*, every element of B appears as the image of at least one element of A.



Onto function

Fig. 4

☛ **REMARKS**

➠ For an onto function
Range = co-domain.

➠ Onto function is also called surjective function.

(d) Into function. A function $f : A \to B$ is called an into function if there is at least one element of the set B, which has no pre-image in the set A. See functions of Fig. 2 and 3.

☛ **REMARK**

➠ In an into function : Range $\subset$ co-domain.

(e) One-one into function. A function $f : A \to B$ is a one-one into function if it is both one-one and into function, *i.e.*, different element in A are joined to different element in B and there are some elements in B, which are not joined to any element in A.

Symbolically. One-one into function is defined as

(i) Range $\subset$ co-domain

(ii) $f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2$

(f) One-one onto function. If a function $f : A \to B$ is both one-one and onto *i.e.*, the different points in A are joined to different points in B and no point in B is left vacant.

☛ **REMARKS**

➠ One-one onto function is also known as bijective function.

➠ For a one-one onto function, Range=co-domain and $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$.

➠ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

(g) Many-one into function. A function $f : A \to B$, which is both many one and into function is called a many one into function, *i.e.*, two or more points in A are joined to some points in B and there are some points in B which are not joined to any point in A.

☛ **REMARK**

➠ For many-one into function, we have

(i) Range $\subset$ Co-domain

(ii) $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$.

(h) Many-one onto function. If a function $f : A \to B$ is both many one and onto function, then it is many-one and onto function, *i.e.*, in B, one point is joined to at least one point in A and two or more points in A are joined to the same points in B.

☛ **REMARK**

➠ For many-one onto function, we have

(i) Range $\subset$ Co-domain

(ii) $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$.

1.7 BOUNDEDNESS OF A SUBSET OF REAL NUMBERS R

1.7.1 UPPER BOUND OF A SUBSET OF R

A subset S of $\boldsymbol{R}$ is said to be bounded above if there exists a real number u such that $s \leq u$, $\forall s \in S$. (MEERUT 1990)

The real number u is said to be upper bound of S.

If there exists no such upper bounds, then the set is said to be unbounded above.

For example:

(i) The set of natural number $\boldsymbol{N} = (1, 2, 3, ...)$ is not bounded above or unbounded above.

(ii) The set of positive integers $\boldsymbol{Z}^+$ is not bounded above.

(iii) The set $S=[1,2,3,4]$ is bounded above by 4.

(iv) The set $\left\{\frac{1}{n}: n \in \boldsymbol{N}\right\}$ is bounded above by 1.

(v) The set of negative integers is bounded above by 0.

1.7.2 LOWER BOUND OF A SUBSET OF R

A subset S of $\boldsymbol{R}$ is said to be bounded below if there exists a real number l such that $s \leq l, \ \forall s \in S$. (MEERUT 2000, 2001)

The real number l is said to be the lower bound of S, if there exists no such lower bound, then the set is said to be unbounded below.

For example :

(i) The set of natural number $\boldsymbol{N}$ is bounded below by 1.

(ii) The set $\left[\frac{1}{n}: n \in \boldsymbol{N}\right]$ is bounded below by 0.

(iii) The set $S=[1,2,3,4]$ is bounded below by 1.

(iv) The set of positive real numbers is bounded below.

1.7.3 BOUNDED SET

A subset S of $\boldsymbol{R}$ is said to be bounded set if it is bounded below as well as bounded above, *i.e.*, if there exist two real numbers l and u such that $l \leq s \leq u, \ \forall s \in S$.

Equivalently, if there exists an interval $I(=[l,u])$ such that $S \subseteq I$.

For example:

(i) Every finite set is bounded.

(ii) The set $\left[\frac{1}{n}: n \in \boldsymbol{N}\right]$ is bounded.

1.7.4 UNBOUNDED SET

A subset S of $\boldsymbol{R}$ which is not bounded is called an unbounded set.

For example:

(i) The sets $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ are unbounded sets.

(ii) Set of all prime numbers is an unbounded set.

1.7.5 LEAST UPPER BOUND (OR SUPREMUM)

A real number u is said to be a least upper bound of a set S if

(i) u is an upper bound of S

(ii) if u' is another upper bound of S, then $u \leq u'$, *i.e.*, no real number less than u can be an upper bound of S.

1.7.6 GREATEST LOWER BOUND (OR INFIMUM)

A real number l is called a greatest lower bound of a set S if

(i) l is a lower bound of S

(ii) if l' is another lower bound of S, then $l' \leq l$, *i.e.*, no

FACTS : TO THE POINT

- If a set is bounded above, then it has infinitely many upper bounds in as much as every number greater than an upper bound is also an upper bound.
- If a set is bounded below, then it has infinitely many lower bounds in as much as every number smaller than a lower bound is also a lower bound.
- It is not necessary that lower bounds and upper bounds of a set S are the members of S.
- The null set ϕ is bounded but it neither possesses lower bound nor upper bound.
- Supremum is defined only for the bounded above sets and infimum for the subset, which are bounded below.
- The supremum and infimum of a set may or may not belong to the set. If supremum of a set belongs to the set, then supremum is the largest element of the set.
- If infimum of a set belongs to the set, then infimum is the smallest element of the set.
- Supremum and infimum of a bounded subset of $\boldsymbol{R}$, are unique.

real number greater than l can be a lower bound of S.

For example : If $S = \left\{\frac{1}{n} : n \in \boldsymbol{N}\right\}$, then *l.u.b.* $= 1$ and *g.l.b.* is 0.

☛ **REMARKS**

- If a real number u is the supremum of a subset S of real numbers then, for every $\varepsilon > 0$, there exists a real number $x \in S$ such that $u - \varepsilon < x < u$.
- If a real number l is the infimum of a subset S of real numbers then, for every $\varepsilon > 0$, there exists a real number $x \in S$ such that $l \le x < l + \varepsilon$.
- In case of singleton set $S = [a], a \in \boldsymbol{R}$, supremum and infimum coincide.
- If u and l are the supremum and infimum of a non-empty subset S of $\boldsymbol{R}$, then $l \le u$.

THEOREM 1. ***The supremum of a set $S \subset \boldsymbol{R}$, if exists, is unique.*** (MEERUT 1990P, 98)

PROOF. Let S be a non-empty subset of $\boldsymbol{R}$.

Let if possible, s_1 and s_2 be two supremum of S.

To show $s_1 = s_2$.

Since we assume that s_1 and s_2 are the supremums of S.

$\Rightarrow$ s_1 and s_2 are the upper bounds of S.

Let, first suppose s_1 is a supremum and s_2 is an upper bound of S, then

$$s_1 \le s_2 \qquad \ldots(1)$$

Now, if s_2 is the supremum and s_1 is the upper bound of S, then

$$s_2 \le s_1 \qquad \ldots(2)$$

From (1) and (2)

$$s_1 = s_2.$$

Hence, supremum of a set, if exists is unique.

THEOREM 2. ***The infimum of a set, if exists, is unique.***

PROOF. Proof is similar as theorem 1 and left to the reader.

THEOREM 3. ***If S be a non-empty subset of $\mathbf{R}$, then a real number s is the supremum of S if and only if***

(i) $x \le s \quad \forall\, x \in S$

and (ii) for each positive real number ε, there exists a real number $x \in S$ such that $x > s - \varepsilon$.

PROOF. (i) **Necessary Condition** (only if part)

Let us first suppose the given condition is necessary for s to be the supremum of the set S.

Let s be the supremum of $S \Rightarrow s$ is an upper bound of S.

By definition $x \le s \ \forall x \in S$.

Let $\varepsilon > 0$ be any real number. Then obviously $s - \varepsilon < s$.

$\Rightarrow$ $(s - \varepsilon)$ is not an upper bound of S. ($\because s$ is l.u.b. of S)

Hence, there must exist some $x \in S$ such that $x > s - \varepsilon$.

(ii) **Sufficient part** (If part)

Let us suppose (i) and (ii) holds.

Then, to show $s = \sup S$.

By condition (i), we have that s is an upper bound of S. To show s is the supremum of S, for this, it is enough to show that no real number less than s can be an upper bound of S.

Let s_1 be any real number less than s

$\Rightarrow \quad s - s_1 > 0$.

Let us take $\varepsilon = s - s_1 \Rightarrow \varepsilon > 0$.

Then by condition (ii), there exists $x \in S$ such that $x > s - \varepsilon$.

$\Rightarrow \quad x > s - (s - s_1) \Rightarrow x > s_1,\ x \in S$.

$\Rightarrow \quad s_1$ is not an upper bound of S.

Hence, we can say that s is an upper bound of S and no real number less than s is an upper bound of S.

$\Rightarrow \quad s$ is the supremum of S.

THEOREM 4. ***Let S be a non-empty subset of $\mathbf{R}$, then a real number t is the infimum of S if and only if***

(i) $x \geq t \quad \forall\, x \in S$

and (ii) for each real number $\varepsilon > 0$, there exists a real number $x \in S$ such that $x > t + \varepsilon$.

PROOF. Proof is similar as theorem 3.

SOLVED EXAMPLES

EXAMPLE 1. ***(i) The set R^+ of positive real numbers, is bounded below and unbounded above.***

(ii) The set R^- of negative real numbers, is bounded above and unbounded below.

SOLUTION. (i) Since every member of $\boldsymbol{R}^- \cup [0]$ is a lower bound of $\boldsymbol{R}^+$, therefore $\boldsymbol{R}^+$ is bounded below.

To prove $\boldsymbol{R}^+$ is unbounded above.

Let if possible, suppose u is an upper bound of $\boldsymbol{R}^+$ we have $u \geq 1$ for $1 \in \boldsymbol{R}^+$. Since $2 \in \boldsymbol{R}^+$, $2 > 0$ and so $u \geq 1, 2 > 0$ gives $u + 2 > 1 + 0$, *i.e.*, $u + 1 > 0$. Thus $(u+1) \in \boldsymbol{R}^+$ and $(u+1) > u$ which is a contradiction, that u is an upper bound of $\boldsymbol{R}^+$.

Hence, $\boldsymbol{R}^+$ is unbounded above.

(ii) Proof follows in a similar manner.

EXAMPLE 2. ***The set R is an unbounded set.***

SOLUTION. From example 1, we conclude that the set $\boldsymbol{R}^+$ is unbounded above and $\mathbf{R}^-$ is always unbounded below.

Also, $\boldsymbol{R} = \boldsymbol{R}^- \cup [0] \cup \boldsymbol{R}^+$

$\Rightarrow \quad \boldsymbol{R}$ is not bounded.

EXAMPLE 3. ***The null set ϕ is neither bounded below or above, nor unbounded.***

SOLUTION. Since, there is no member in ϕ, we can not check whether a given real number can be a bound for ϕ or not. Thus, bounds for ϕ do not exist. On the other hand, we can as well say that every real number is a lower or upper bound for there is no member in ϕ which does not satisfy the required property of bounds.

EXAMPLE 4. ***Show that every non-empty finite subset of R is bounded.***

SOLUTION. Let S be a non-empty finite subset of $\boldsymbol{R}$.

$\Rightarrow \quad S$ contains a finite number of elements. Then by the properties of the ordered relation in $\boldsymbol{R}$, out of these elements one element $a \in S$ shall be the smallest element of S and another element $b \in S$ shall be the greatest element of S.

$\Rightarrow \quad a \le x \le b \ \ \forall\, x \in S$

Hence, S is always bounded.

EXAMPLE 5. ***Find the supremum and infimum of the set*** $S = \{x \in Z : x^2 \le 25\}$. (MEERUT 2001)

SOLUTION. Since $S = \{x \in Z : x^2 \le 25\}$

$= \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

Since S is a finite subset of R, the smallest member of S is -5, which is a lower bound of S, and hence infimum of S is -5. Similarly 5 is the supremum of S.

EXAMPLE 6. ***Find the supremum and infimum, if they exist, of the following sets***

(i) $\left\{\frac{1}{n} : n \in N\right\}$ (MEERUT 1994, 95, 97, 2000, 2001)

(ii) $\left\{x \in Q : x = \frac{n}{n+1},\ n \in N\right\}$ (MEERUT 1994)

(iii) $\left\{1 + \frac{(-1)^n}{n} : n \in N\right\}$ (MEERUT 1990S)

(iv) $\left\{\pi + \frac{1}{2}, \pi + \frac{1}{4}, \pi + \frac{1}{8}, \ldots\right\}$ (MEERUT 1993, 93S)

SOLUTION. (i) Here, we have

$$S = \left\{\frac{1}{n} : n \in N\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$$

The set is bounded above by 1, also any member less than 1 is not an upper bound of S, therefore $\sup S = 1$

Also, 0 is a lower bound of S, because $x \ge 0,\ \forall x \in S$. Let l be any arbitrary positive small number, then there exists $n \in N$ such that $\frac{1}{n} < l$, which shows that l is not an upper bound of S. Thus 0 is a lower bound of S and no other positive real number is a lower bound of S. Therefore, infimum of $S = 0 \notin S$.

(ii) Let $S = \left\{\frac{n}{n+1} : n \in N\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

Then, the set is bounded below by $\frac{1}{2}$ and any number greater than $\frac{1}{2}$ can not be a lower bound of S. Therefore infimum of $S = \frac{1}{2}$.

Also, $\left(\frac{n}{n+1}\right) < 1,\ \forall n \in N$, therefore 1 is an upper bound of S, and any number less than 1 not be an upper bound of S. Therefore, supremum of $S = 1$.

(iii) Let $S = \left\{1 + \frac{(-1)^n}{n} : n \in N\right\} = \left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, \ldots\right\}$

$$= \left\{\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \ldots, \frac{2n-2}{2n-1} \ldots\right\} \cup \left\{\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \ldots, \frac{2n+1}{2n} \ldots\right\}.$$

Here, we have that the proper fraction $\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$ are increasing and tending to 1, and the improper fractions begin with $\frac{3}{2}$ are decreasing and tending to 1.

Therefore, infimum of $S = 0$ and supremum of $S = \frac{3}{2}$.

(iv) Let $S = \left\{\pi+\frac{1}{2},\ \pi+\frac{1}{4}, \pi+\frac{1}{8}, \ldots\right\}$

Here, we have $x \leq \pi+\frac{1}{2}\ \forall\, x \in S$

$\Rightarrow$ $\pi+\frac{1}{2}$ is an upper bound for S.

Since, $\pi+\frac{1}{2} \in S$, therefore no real number less than $\pi+\frac{1}{2}$ can be an upper bound for S. Thus, $\pi+\frac{1}{2}$ is the least upper bound. Therefore, supremum of $S = \pi+\frac{1}{2}$.

Similarly, we can show that π is the infimum of S.

1.8 SEQUENCES

Let $\boldsymbol{N}$ be the set of natural numbers and S be any set of real numbers. A function, whose domain is the set of natural number s and range is a subset of S, is called a sequence in S.

Symbolically. If we define a function $f : \boldsymbol{N} \to S$, then f is a sequence. We shall denote a sequence in a number of ways:

(i) Usually, a sequence is denoted by its images. For a sequence f, the image corresponding to $n \in \boldsymbol{N}$ is denoted by f_n or $<f(n)>$ and is called the n^{th} term of the given sequence. For example $<1,4,9,\ldots>$ is the sequence whose n^{th} term is n^2.

(ii) Using in order, the first few element of a sequence, the rule for writing down different elements becomes clear. For example, $<1, 2, 3, \ldots>$ is the sequence whose n^{th} term is n.

(iii) Defining a sequence by a recurrence formula *i.e.*, by a rule which express the n^{th} term by $(n-1)^{\text{th}}$ term. For example, let $a_1 = 1, a_{n+1} = 2a_n \quad \forall n \geq 1$.

These above relations define a sequence whose n^{th} term is 2^{n-1}.

☛ REMARKS

- A sequence is represented as $< s_n >$ or $\{s_n\}$, when s_n is the n^{th} term of the sequence.
- The set of all distinct terms of a sequence is called the range set of that sequence.
- A sequence, whose range is a subset of $\boldsymbol{R}$ is called a real sequence or a sequence of real numbers.

Constant sequence. A sequence $< s_n >$ defined by $s_n = a,\ \forall n \in \boldsymbol{N}$, is called a constant sequence.

Equality on sequence. Two sequence $< s_n >$ and $< t_n >$ are said to be equal, if $s_n = t_n \quad \forall n \in \boldsymbol{N}$.

1.8.1 OPERATION ON SEQUENCES

Since, the sequence are real valued functions, therefore, the sum, difference, product etc. of two sequence are defined as follows:

(i) If $< s_n >$ and $< t_n >$ be any two sequence, then the sequence, whose n^{th} terms are $s_n + t_n$, $s_n - t_n$ and $s_n . t_n$ are respectively known as the sum, difference and product of the sequence $< s_n >$ and $< t_n >$ and are denoted by $< s_n + t_n >, < s_n - t_n >$ and $< s_n . t_n >$ respectively.

(ii) If $s_n \neq 0 \quad \forall n \in \boldsymbol{N}$, then the sequence, whose n^{th} term is $\frac{1}{s_n}$ is called reciprocal of the sequence $< s_n >$ and is denoted by $< \frac{1}{s_n} >$.

(iii) The sequence , whose n^{th} term $s_n / t_n (t_n \neq 0 \quad \forall n \in \boldsymbol{N})$ is known as the quotient of the sequence $< s_n >$ by the sequence $< t_n >$ and is denoted by $< \frac{1}{s_n} >$.

(iv) The sequence, whose n^{th} term is ks_n, where $k \in \boldsymbol{R}$ is known as the scalar multiple of the sequence $< s_n >$ by k and is denoted by $< ks_n >$.

1.8.2 BOUNDED SEQUENCE

(i) Bounded below sequence. A sequence $< s_n >$ is said to be bounded below is there exists a real number l such that $s_n \geq l \quad \forall n \in \boldsymbol{N}$..

The number l is known as the lower bound of the sequence $< s_n >$.

(ii) Bounded above sequence. A sequence $< s_n >$ is said to be bounded above if there exists a real number u such that $s_n \leq u \quad \forall n \in \boldsymbol{N}$.

The number u is said to be upper bound of the sequence $< s_n >$.

(iii) Bounded sequence. A sequence $< s_n >$ is said to be bounded if it is bounded above as well as bounded below.

(iv) Unbounded sequence. A sequence $< s_n >$ is said to be unbounded if it is not bounded.

(v) Least upper bound. If a sequence $< s_n >$ is bounded above, then there exists a number u_1 such that

$$s_n \leq u_1 \quad \forall n \in \boldsymbol{N}. \qquad \ldots(1)$$

This number u_1 is called an upper bound of the sequence $< s_n >$. If $u_1 < u_2$. Then from (1), we find that

$$s_n < u_2 \quad \forall n \in \boldsymbol{N}$$

which implies, u_2 is also an upper bound of the sequence $< s_n >$. Hence, we can say that any number greater than u_1 is an upper bound of $< s_n >$.

Hence, a sequence has an infinite number of upper bounds, if it is bounded above. Let u be the least of all the upper bound of the sequence $< s_n >$. Then u is defined as the least upper bound (l.u.b) or supremum of the sequence $< s_n >$.

(vi) Greatest lower bound. If a sequence $< s_n >$ is bounded below, then there exists a number $l_1 \in \boldsymbol{R}$ such that

$$l_1 \leq s_n \quad \forall n \in \boldsymbol{N} \qquad \ldots(2)$$

This number l_1 is known as the lower bound of $< s_n >$. If $l_2 < l_1$, then from (2)

$$l_2 \leq s_n \quad \forall n \in \boldsymbol{N}$$

which implies, l_2 is also a lower bound of the sequence $< s_n >$. Hence, we can say any number less than l_1 is a lower bound of $< s_n >$.

Hence, a sequence has infinite number of lower bounds, if it is bounded below. Let l be the greatest of

FACTS : TO THE POINT

- If $c > 0$ then $<a_n>$ converges to a if and only if for each $\varepsilon > 0$ there exists a positive integer m, depending on ε such that $|a - a_n| \leq c\varepsilon \; \forall \; n \geq m$
- If $<a_n>$ is a convergent sequence in R such that $\lim_{n\to\infty} a_n > r$ then $a_n > r$ $\forall$ sufficiently large n, $r \in \boldsymbol{R}$.
- Let $r \in \boldsymbol{R}$ and $<a_n>$ be a convergent sequence in $\boldsymbol{R}$ such that $a_n \geq r$ for all sufficiently large n. Then $\lim_{n\to\infty} a_n \geq r$.
- If $r > 1$ then $r^n \to \infty$ as $n \to \infty$.
- If $a > 1$ then $\log_a n \to \infty$ as $n \to \infty$.
- An increasing sequencing of non-negative real numbers diverges to infinitely if and only if it is not bounded above.
- The intersection of a nested sequence of closed intervals in $\boldsymbol{R}$ is non-empty.
- Let $<s_n>$ be a sequence of real numbers then atleast one of the following hold:
  - (i) $<s_n>$ has a convergent subsequence
  - (ii) $<s_n>$ has a strictly increasing subsequence
  - (iii) $<s_n>$ has a strictly decreasing subsequence

all the lower bounds of the sequence $< s_n >$. Then l is known as greatest lower bound (g.l.b.) or infimum of the sequence $< s_n >$.

ILLUSTRATIONS

- The sequence $< n^2 >$ is bounded below by 1 but not bounded above.
- The sequence $< \frac{n}{n+1} >$ is bounded as $\frac{1}{2} \le \frac{n}{n+1} < 1 \quad \forall n \in \boldsymbol{N}$.
- The sequence $< \frac{1}{n} >$ is bounded since $\left|\frac{1}{n}\right| \le 1 \quad \forall n \in \boldsymbol{N}$.
- The sequence $< 2^n >$ is bounded below and has smallest term as 2. Every member of $]-\infty, 2]$ is a lower bound of the sequence and the sequence is not bounded above.

1.8.3 LIMIT POINT OF THE SEQUENCE

A real number l is called a limit point of a sequence $< s_n >$ if every *nbd* of l contains infinite number of terms of the sequence.

Thus, $l \in \boldsymbol{R}$ is a limit point of the sequence $< s_n >$ if for given $\varepsilon > 0, s_n \in]l-\varepsilon, l+\varepsilon]$ for infinitely many points.

Here it must be noted that

(i) Limit point of a sequence need not be a member of the sequence.

(ii) A limit point of a sequence may or may not be a limit point of the range of the sequence but the limit point of the range of a sequence is always a limit point of the sequence.

(iii) In case of real number, limit points of a sequence may also be called accumulation, cluster or condensation points.

ILLUSTRATIONS

- The sequence $< \frac{1}{n} >$ has one limit point, *i.e.*, 0.
- The sequence $< (-1)^n >$ has two limit points 1 and -1.
- The sequence $< n >$ has no limit point.
- The sequence $< 1 + \frac{(-1)^n}{n} >$ has one limit point, *i.e.*, 1.

Sufficient Conditions for number l to be or not be a limit point of the sequence $< s_n >$:

(i) If for every $\varepsilon > 0$, $\exists m \in \boldsymbol{N}$ such that $s_n \in]l-\varepsilon, l+\varepsilon[\quad \forall n \ge m$ or equivalently $|s_n - l| < \varepsilon \quad \forall n \ge m$, then l is the limit point of the sequence $< s_n >$.

(ii) If for any $\varepsilon = 0, s_n \in]l-\varepsilon, l+\varepsilon[$ for only a finite number of values of n, then l is not a limit point of the sequence $< s_n >$. Such a condition is also necessary for a number l not to be limit point of the sequence $< s_n >$.

1.8.4 BOLZANO-WEIERSTRASS THEOREM FOR SEQUENCE (KANPUR 2000)

STATEMENT. ***Every bounded sequence has at least one limit point.***

PROOF. Let $S = \{s_n : n \in \boldsymbol{N}\}$ be the range set of the bounded sequence $< s_n >$.

Then, S is a bounded set. Now, there may be two cases :

(i) Let S be a finite set. Then $s_n = p$ for infinitely many indices n. Here $p \in \boldsymbol{R}$. Obviously p is a limit point of $< s_n >$.

(ii) Let S be an infinite set. Since, S is bounded, then by Bolzano-Weirstrass theorem for set of real numbers S has a limit point say p. Therefore, every nbd of p contains infinity many distinct points of S, *i.e.*, infinitely many terms of $< s_n >$ and hence p is a limit point of the sequence $< s_n >$.

1.8.5 LIMIT SUPERIOR AND LIMIT INFERIOR

The greatest limit point of a bounded sequence is called the upper limit or limit superior and is denoted by $\overline{\lim}\, s_n$ and the smallest limit point of a bounded sequence is called the lower limit or limit inferior and is denoted by $\lim s_n$

- By definition, it is obvious that $\lim s_n \leq \overline{\lim}\, s_n$.
- A bounded sequence $< s_n >$ for which the upper limit and lower limit coincide with real number l is said to converge to l.

Limit of a sequence. A sequence $< s_n >$ is said to have a limit l if for a given $\in > 0$ $\exists$ a positive integer m such that $|s_n - l| < \in \quad \forall n \geq m$.

1.8.6 CONVERGENT SEQUENCE

A sequence $< s_n >$ is said to converge to a number l, if for a given $\in > 0$ there exists a positive integer m such that $|s_n - l| < \in \quad \forall n \geq m$.

☛ REMARK

➠ A sequence $< s_n >$ is said to be convergent iff it is bounded and has one and only one limit point.

1.8.7 DIVERGENT SEQUENCE

A sequence, which is not convergent, is known as divergent sequence.

1.8.8 OSCILLATORY SEQUENCE

A sequence $< s_n >$ is said to be an oscillatory sequence if it is neither convergent nor divergent.

An oscillatory sequence is said to be oscillate finitely or infinitely according as it is bounded or unbounded.

In other words, we can say

(i) A bounded sequence, which is not convergent is said to be oscillate finitely.

(ii) An unbounded sequence, which does not diverge, is said to be oscillate infinitely.

(iii) A bounded sequence, which does not converge and has at least two limit points is said to be oscillate finitely.

✍ ILLUSTRATIONS

- The sequence $\langle 1+(-1)^n \rangle$ oscillate finitely.
- The sequence $\langle (-1)^n \rangle$ oscillate finitely.
- The sequence $< (-1)^n \left(1+\frac{1}{n}\right) >$ oscillate finitely.
- The sequence $< n(-1)^n >$ oscillate infinitely.

1.9 COUNTABLE SET

A set which can be put in one to one correspondence with the set of all natural numbers or with its subset is said to be a countable set. (MEERUT 2003,04; KANPUR-2000)

For example: If $A=\{2,4,6,8,...\}$, then $\exists$ an one to one mapping $f: A \to \mathbf{N}$ s.t. $f(x)=x/2$. Then A is a countable set.

Further if $B=\{2,4,6,8,10\}$, then this set can be put in one to one correspondence with the set $A=\{1,2,3,4,5\}$, which is a subset of $\boldsymbol{N}$.

Obviously a countable set can be finite or infinite both.

An infinite countable set is also called as countable infinite or enumerable set or denumerable set. (MEERUT 2001)

Thus enumerable set always means an infinite countable set.

Further note that if a set 'A' is countable then its elements can be put in one to one correspondence with set $\boldsymbol{N}=\{1,2,3,...\}$. If we denote the elements of A corresponding to the natural numbers $1,2,3,...$, by $a_1,a_2,a_3,...$ etc., then the set A can be written as $A=\{a_1,a_2,a_3,...\}$.

Thus a set is countable iff its elements can be written in the form of a sequence.

☛ **REMARK**

➠ A set which is not countable, is said to be uncountable.

THEOREM 1. ***Every subset of a countable set is countable.***

(MEERUT 1987,88,94,2003(BP); KANPUR 2000)

PROOF. Let A be a countable set, then A can be written as a sequence. Let $A=\{a_1,a_2,a_3,...\}$. If A is finite then its every subset will also be finite and so will be countable. Now consider that A is enumerable. Let B be any subset of A.

If B is finite or empty then the result is trivial, so let $B \neq \phi$ and let k_1 be the least positive integer *s.t.* $a_{k_1} \in B$. Again, let k_2 be the least positive integer with $k_2 > k_1$ *s.t.* $a_{k_2} \in B$. Dealing with the elements of B in this way, we reach to the conclusion that B can be written as $\{a_{k_1},a_{k_2},...\}$ which is a sequence and hence B is a countable set.

THEOREM 2. ***The union of enumerable collection of enumerable sets is also enumerable.***

(MEERUT 1972,84; GARHWAL 1991,95; KANPUR 2000)

PROOF. Let $A=\{A_1,A_2,...,\}$ be an enumerable collection of enumerable sets where $A_i=\{a_{i_1},a_{i_2},a_{i_3},...\}$, $\forall i \in \boldsymbol{N}$ denote enumerable sets. For the proof of the theorem, we shall construct a progression in which all the elements of A appear. Such a list is as follows :

$$\bigcup_{i\in N} A_i = \{a_{11} \to a_{12} \to a_{13} \to a_{14} \cdots$$
$$a_{21} \quad a_{22} \quad a_{23} \quad a_{24} \cdots$$
$$a_{31} \quad a_{32} \quad a_{33} \quad a_{34} \cdots$$
$$a_{41} \quad a_{42} \quad a_{43} \quad a_{44} \cdots$$

Choose the path as shown above; then every element of A lies somewhere on the path (*i.e.*, each element occupies some particular position say r^{th} position $r \in \boldsymbol{N}$) and hence a one-one correspondence between $\cup A_i$ and $\boldsymbol{N}$ is implied showing thereby that A is enumerable. Thus we can write $\cup A_i=\{a_{11},a_{12},a_{21},a_{31},a_{22}...\}$ as a sequence.

$\Rightarrow$ set $\cup A_i$ is enumerable.

Besides the above argument, note that the mapping $f: \bigcup_{i\in N} A_i \to \boldsymbol{N}$ given by

$$f(a_{pq})=\frac{(p+q-2)(p+q-1)}{2}+p \quad (=say\ \ n \in \boldsymbol{N})$$

shows the enumeration of

$$\cup A_i=\{a_{11},a_{21},a_{12},a_{13},a_{22},a_{31},a_{14},a_{23},a_{32},a_{41},...\}.$$

i.e., this mapping assigns a unique place to each element of the set $\cup A_i$ in the above sequence showing that $\cup A_i$ is countable.

COR.1. ***The union of countable collection of countable sets is also countable.***

(MEERUT 1986,87,91,94,2001; AMRAVATI 1997; GARHWAL 2001; KANPUR 2001,02)

COR.2. ***If A_i is countable infinite set, then prove that $\bigcup_{i=1}^{n} A_i$ is also countable infinite.*** (GARHWAL 1995)

THEOREM 3. ***The set $N \times N$ is enumerable.*** (MEERUT 1983,98, GARHWAL 2003)

PROOF. Here note that $\boldsymbol{N} \times \boldsymbol{N} = \{(u,v) : u,v \in \boldsymbol{N}\}$; then clearly $\boldsymbol{N} \times \boldsymbol{N}$ can be arranged as shown below :

$$\begin{aligned}\boldsymbol{N} \times \boldsymbol{N} = \{&(1,1),(1,2),(1,3),(1,4)\ldots\\ &(2,1),(2,2),(2,3),(2,4)\ldots\\ &(3,1),(3,2),(3,3),(3,4)\ldots\\ &\ldots \quad \ldots \quad \ldots \quad \ldots\} = \cup A_i,\end{aligned}$$

where $A_i = \{(i,n) : n \in \boldsymbol{N}\}$

Now obviously each $A_i (i = 1,2,3,\ldots)$ is enumerable which shows that the set $\boldsymbol{N} \times \boldsymbol{N}$, union of enumerable collection of enumerable sets is also enumerable by Theorem 2.

THEOREM 4. ***The set Q of all rational numbers is enumerable.***

(MEERUT 1984,90,95,97,2001(BP), KANPUR 1996)

PROOF. Recall that a rational number is written as $p/q : p,q \in \boldsymbol{Z}$ and $q \neq 0$.

Let $A_q = \{p/q : p,q \in \boldsymbol{Z}\}$. Then obviously A_q is equivalent to $\boldsymbol{Z}$ and we know that $\boldsymbol{Z}$ is enumerable and so A_q is enumerable. Also the collection $\boldsymbol{Q} = \bigcup_{q \in \boldsymbol{Z}_0} A_q$ is enumerable where $\boldsymbol{Z}_0$ is a subset of $\boldsymbol{Z}$, because $\boldsymbol{Q}$ is expressed as enumerable union of enumerable sets.

THEOREM 5. **(a)** ***The set of all real numbers in the open interval $(0,1)$ is not enumerable.***

(MEERUT 1989,96; HIMACHAL 2000; KANPUR 2004)

(b) ***Prove that the set of all real numbers in the closed interval $[0,1]$ is not enumerable.*** (MEERUT 1998,2004; GARHWAL 1995,97,2003; HIMACHAL 2002)

PROOF. (a) Suppose contradiction, *i.e.*, let the given interval be enumerable. Then the elements of the interval can be written as a sequence $\{x_1, x_2, x_3, \ldots\}$. Now using the decimal expansion of these x_m's, we can get as follows :

$$\begin{aligned}x_1 &= .a_{11}a_{12}a_{13}\ldots a_{1n}\ldots\\ x_2 &= .a_{21}a_{22}a_{23}\ldots a_{2n}\ldots\\ x_3 &= .a_{31}a_{32}a_{33}\ldots a_{3n}\ldots\\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\ x_n &= .a_{n1}a_{n2}a_{n3}\ldots a_{nm}\ldots\\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\end{aligned}$$

where a_{ij}'s may be any integer from 0 to 9. Now let

$$x_n = b_1 b_2 b_3 \ldots$$

where $b_1, b_2, \ldots$ are the digits from 0 to 9 *s.t.* $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}$ etc.

In general $b_m \neq a_{mm}, \forall m \in \boldsymbol{N}$.

Hence, $x \neq x_1, x \neq x_2, \ldots, x \neq x_m, \ldots$.

Obviously, $x \in (0,1)$ and $x \notin \{x_1, x_2, ...\}$; thus we set that besides countable set $\{x_1, x_2, ...\}$ there exists element belonging to interval $(0,1)$, showing that the set of all real numbers lying in the interval $(0,1)$ is not enumerable.

(b) Open interval (0, 1) is a subset of [0, 1]. Since (0, 1) is not denumerable, the interval [0, 1] is also not denumerable.

THEOREM 6. ***The set of all real numbers is not enumerable, i.e., R is not enumerable.***

(MEERUT 1991,97; GARHWAL 1996; KANPUR 1996,2000,02)

PROOF. Suppose contradiction, *i.e.*, let $\boldsymbol{R}$ be enumerable. Then its every subset must be enumerable (Theorem 1). Consider the set of all real numbers lying in the interval (0, 1) which is not enumerable by preceding theorem 5; which is again a contradiction since it is the subset of $\boldsymbol{R}$. Hence the theorem.

THEOREM 7. ***The set of all irrational numbers is not enumerable.***

PROOF. Let the contrary be true. Also by theorem 4, the set of all rational numbers is enumerable. Thus union of rational and irrational numbers would also be enumerable. But we know that $\boldsymbol{R}$ is the union of rational and irrational numbers and hence $\boldsymbol{R}$ would be enumerable which is a contradiction according to Theorem 6. Hence the set of all irrational numbers is not enumerable.

1.10 ALGEBRAIC NUMBERS

Let

$$P_n(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + ... + \alpha_1 x + \alpha_0 (\alpha_n \neq 0)$$

be a polynomial with α_i's as integral numbers, then we define the algebraic number as a root of the polynomial equation $P_n(x) = 0$ of the above form.

THEOREM 1. ***The set of all algebraic numbers is enumerable.***

(MEERUT 1996; GARHWAL 1993; KANPUR 1972)

PROOF. Consider the algebraic equation, $\alpha_0 x^n + \alpha_1 x^{n-1} + ... + \alpha_n$ of degree n with $\alpha_0 \neq 0$. Now we define the rank of this equation :

$$|\alpha_0| + |\alpha_1| + |\alpha_2| + ... + |\alpha_n| = m .$$

Clearly, rank is a positive number. Also α_i's are integers; so rank is an integer ≥ 1. Obviously for a given rank the roots of the equation will be finite and therefore will be enumerable.

Again we can put a one-one correspondence in the set of natural numbers with the algebraic equation arranged with respect to rank and hence the set of all algebraic equations is enumerable. Now each algebraic equation has enumerable number of roots and so the set of all algebraic numbers is the enumerable collection of enumerable sets and hence enumerable by Theorem 2.

SOLVED EXAMPLES

EXAMPLE 1. ***Show that the set P of all polynomials***

$$\boldsymbol{P_n(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + ... + \alpha_1 x + \alpha_0, (\alpha_n \neq 0)}$$

with integral (rational) coefficients is denumerable. (GARHWAL 2001)

SOLUTION. If $|\alpha_n|+|\alpha_{n-1}|+\ldots+|\alpha_0|=m$, then for each pair of natural numbers (m,n), set P_{mn} of all the polynomials of the form

$$P_n(x)=\alpha_n x^n+\alpha_{n-1}x^{n-1}+\ldots+\alpha_1 x+\alpha_0$$

is finite and hence countable.

Also the sets $P_{(m,n)}=P_K,\ K=(m,n)\in \boldsymbol{N}\times\boldsymbol{N}$ themselves are countable.

Therefore the set $P=\bigcup_{(m,n)\in \boldsymbol{N}\times\boldsymbol{N}} P_{m,n}$ is also countable.

1.11 CARDINALLY EQUIVALENT SETS

A set P is said to be cardinally equivalent to a set Q, if there exists at least one one-to-one map from P to Q and is denoted by $P \sim Q$.

THEOREM 1. ***Prove that the relation $P \sim Q$ in the family of sets is an equivalence relation.***

PROOF. Recall that a relation R in a set is an equivalence relation in P, iff

(i) R is reflexive, *i.e.*, $aRa,\ \forall a\in P$

(ii) R is symmetric, *i.e.*, $aRb\Rightarrow bRa,\ \forall a,b\in P$,

(iii) R is transitive, *i.e.*, $aRb, bRc\Rightarrow aRc\ \ \forall a,b,c\in P$.

Here we observe that the given relation is

(i) Reflexive. Since the identity map $\boldsymbol{I}_P: P\to P$ given by $\boldsymbol{I}_P(a)=a, \forall a\in P$ is one-one onto, *i.e.*, $P \sim P$ for every set P.

(ii) Symmetric. $P\sim Q\Rightarrow \exists$ a one-one map $f: P\xrightarrow{onto} Q$

$f^{-1}: Q\xrightarrow{onto} P$ is also one-one $\Rightarrow Q\sim P$.

(iii) Transitive.

$P\sim Q\Rightarrow \exists$ a one-one map $f: P\xrightarrow{onto} Q$

$Q\sim R\Rightarrow \exists$ a one-one map $g: Q\xrightarrow{onto} R$

$\Rightarrow g\ o\ f: P\xrightarrow{onto} R$ is also one-one

$\Rightarrow P\sim R$

Thus the given relation is an equivalence relation. It is to be noted that an equivalence relation decomposes the set P into equivalence classes, any two of which are either equal or mutually disjoint. The set of these mutually disjoint equivalence classes is said to be the quotient set of the set for the given equivalence relation. The equivalence relation is also known as equipollent relation.

SOLVED EXAMPLES

EXAMPLE 1. ***If A and B are the sets of all the real numbers of the intervals $[a_1,a_2]$ and $[b_1,b_2]$ respectively, then show that $A\sim B$.***

SOLUTION. Consider a mapping $f: A\to B$ *s.t.*

If $x\in A$, then

$$f(x)=\frac{b_2-b_1}{a_2-a_1}x-a_1\frac{b_2-b_1}{a_2-a_1}+b_1$$

$$=\left(\frac{b_2-b_1}{a_2-a_1}\right)(x-a_1)+b_1.$$

Mapping f is one-one. For if $f(x)=f(y); x,y\in A$, then

$$\frac{(b_2-b_1)}{(a_2-a_1)}(x-a_1)+b_1=\frac{(b_2-b_1)}{(a_2-a_1)}(y-a_1)+b_1$$

$$\Rightarrow \qquad \left(\frac{b_2-b_1}{a_2-a_1}\right)(x-a_1)=\left(\frac{b_2-b_1}{a_2-a_1}\right)(y-a_1)\Rightarrow x=y$$

$\Rightarrow$ f is one-one.

Mapping f in onto. Let $z\in B\Rightarrow b_1\le z\le b_2$.

If $z=\left(\frac{b_2-b_1}{a_2-a_1}\right)(x-a_1)+b_1$, then $x=\frac{(z-b_1)(a_2-a_1)}{(b_2-b_1)}+a_1$.

Obviously $a_1\le x\le a_2$ and hence $x\in A$.

Thus for any $z\in B$, $\exists x\in A$ *s.t.* $f(x)=z$ and hence f is onto.

$\Rightarrow$ f is one-one onto and hence $A\sim B$.

REMARK

➠ If $A=(a_1,a_2)$, $B=(b_1,b_2)$ then $A\sim B$.

1.12 CARDINAL NUMBER OF A SET

As we have already seen that the relation of equivalence decomposes any collection of sets into equivalence classes containing the equivalent sets. Each equivalent class has a cardinal number which we shall use to represent the property the equivalent sets have in common. It will be in some sense a measure of the points in sets. It is denoted by the card X, *i.e.*, cardinal number of X. The basic property of cardinal number is that card X = card Y iff $X\sim Y$.

Definition. *Any set which is equivalent to the set* {1, 2, 3, ..., *n*} *is said to have the cardinal number n.*

In this way we have defined the cardinal number of the finite sets. The null set is said to have the cardinal number zero. Obviously for finite sets, cardinal number is just the number of element in the sets. (MEERUT 2003,04; GARHWAL 2002)

The cardinal number of $\boldsymbol{N}$ (set of natural numbers) is denoted by $\boldsymbol{\alpha}$ and hence all enumerable sets will have the cardinal number $\boldsymbol{\alpha}$.

The cardinal number of the set of real numbers is denoted by $\boldsymbol{c}$ and all the sets equivalent to the set $\boldsymbol{R}$ are said to have the cardinal number $\boldsymbol{c}$. The set of all real numbers in the interval [0, 1] also has the cardinal number $\boldsymbol{c}$. Since we have proved that all intervals (open or closed) are equivalent to [0,1], hence every interval also has cardinal number $\boldsymbol{c}$. The cardinal number of the set of all real-valued functions defined in the interval [0, 1] is denoted by $\boldsymbol{f}$.

The cardinal number of an infinite set is called a transfinite cardinal number $\boldsymbol{\alpha}$ is considered to be the first (smallest) transfinite cardinal.

Since every finite set has an enumerable subset which is equivalent to be the set of natural number $\boldsymbol{N}$ and so every infinite set has a subset with cardinal number $\boldsymbol{\alpha}$.

Note that the set of cardinal number $\{0,1,2,3,...,\boldsymbol{\alpha},\boldsymbol{c},\boldsymbol{f}\}$ is a superset of set of all natural numbers.

1.12.1 SUM OF CARDINAL NUMBERS

Let A and B be any two disjoint sets; then

$$\text{card } A+\text{card } B=\text{card } A\cup B.$$

In general $\sum_{\alpha\in\Delta} card A_\alpha = card\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)$

where $A_\alpha\cap A_\beta=\phi$, $\forall\alpha,\beta,\in\Delta$ (index set) such that $\alpha\neq\beta$.

1.12.2 PRODUCT OF CARDINAL NUMBERS

We define the product of two cardinal numbers as card $P\times$ card $Q=card(P\times Q)$ where P,Q are any sets.

In general, card $P_1\times$ card $P_2...=$ card [Cartesian product of sets $P_i(i=1,2,3,...)$]

1.12.3 COMPARISON OF CARDINAL NUMBERS

Let P and Q be any two sets, then

(i) card $P <$ card Q, if $\exists$ a set $R \subset Q$ *s.t.* $P \sim R$ and if $P \sim Q$ then card $P =$ card Q.

(ii) card $P \leq$ and Q, if $\exists$ a set $R \subseteq Q$ *s.t.* $P \sim R$, *i.e.*, there exists a one-one map

$$f : P \xrightarrow{onto} R \subseteq Q .$$

(iii) Evidently, card $(P \cup Q) \geq$ card P or card Q.

(iv) $\boldsymbol{\alpha}$ is the smallest infinite cardinal number.

THEOREM 1. ***Prove that card $(P \times Q)$ = cardQ + cardQ + ...card P terms.***

PROOF. We have $P \times Q = \{(x,y) : x \in P, y \in Q\} = \{ \bigcup_{x \in P} (x,y) : y \in Q\}$.

$$\therefore \quad card(P \times Q) = card\left(\{ \bigcup_{x \in P} (x,y) : y \in Q\}\right). \qquad \ldots(1)$$

Let $x \in P$ be arbitrary but fixed, Consider the map

$$f : Q \to \{(x,y) : y \in Q\}$$

defined by $f(y) = (x,y) \quad \forall y \in Q$.

This show that f is one-one. Therefore

$$cardQ = card((x,y) : y \in Q))$$

Observing (1) we get the required result.

EXAMPLE 1. ***Show that $\alpha \leq$ cardP $\leq \alpha \Rightarrow$ card P $= \alpha$***

SOLUTION. $\alpha \leq cardP \Rightarrow \exists$ a set Q such that card $Q \leq card\ P$ where card $Q = \alpha$

$\Rightarrow Q$ is equivalent to a subset of P ...(1)

Again $card\ P \leq \alpha \Rightarrow cardP \leq Q$

$\Rightarrow P$ is equivalent to a subset of Q ...(2)

Combining (1) and (2), we get

$$card\ P = card\ Q \text{ or } cardP = \alpha \qquad (\because card\ Q = \alpha)$$

THEOREM 2. ***Let P and Q be any two sets; then show that***

(i) card P + card Q is unique,

(ii) card P.card Q is unique.

PROOF. Let $P \sim P_1, Q \sim Q_1, P_1 \cap Q_1 = \phi$. Now

(i) $P \sim P_1 \Rightarrow \exists$ a one-one map $f : P \xrightarrow{onto} P_1$

$Q \sim Q_1 \Rightarrow \exists$ a one-one $g : Q \xrightarrow{onto} Q_1$.

Define a function $\psi : P \cup Q \to P_1 \cup Q_1$ *s.t.*

$$\psi(x) = \begin{cases} f(x) & , \ \forall x \in P, \\ g(x) & , \ \forall x \in Q. \end{cases}$$

Evidently ψ is one-one onto since f and g are one-one onto. Hence $P \cup Q \sim P_1 \cup Q_1 \Rightarrow card\ (P \cup Q) = card\ (P_1 \cup Q_1)$

This shows that $card\ P + card\ Q$ is unique.

(ii) $x \in P, y \in Q \Rightarrow (x,y) \in P \times Q$

$$\Rightarrow (f(x), g(y)) \in P_1 \times Q_1 .$$

Define a function $\psi : P \times Q \to P_1 \times Q_1$ *s.t.*,

$$\psi(x.y) = (f(x), g(y)) \quad \forall (x,y) \in P \times Q .$$

Again f and g one-one onto, therefore ψ is one-one onto.

$$\Rightarrow \quad P \times Q \sim P_1 \times Q_1 \Rightarrow card(P \times Q) = card(P_1 \times Q_1) .$$

This shows that card P. card Q is unique.

EXAMPLE 2. ***Show that $(P \times Q) \sim (Q \times P)$ i.e., $(P \times Q)$ is cardinally equivalent to $(Q \times P)$.***

SOLUTION. We have $(P \times Q) = \{(p,q) : p \in P, q \in Q\}$

Define a map $f : (P \times Q) \to (Q \times P)$ by the formula $f(p,q) = (q,p)$.

f is one-one. Let $f((p_1,q_1)) = f((p_2,q_2))$

$\Rightarrow \quad (q_1,p_1) = (q_2,p_2) \quad \Rightarrow \quad q_1 = q_2$ and $p_1 = p_2$

$\Rightarrow \quad (p_1,q_1) = (p_2,q_2)$.

Obviously f is onto.

Hence, $(P \times Q) \sim (Q \times P)$.

THEOREM 3. ***Let card $P = p$, card $Q = q$, card $R = r$, then show that :***

(i) $p + q = q + p$,

i.e., addition of cardinal numbers is commutative.

(ii) $p \,.\, q = q \,.\, p$ (GARHWAL 2002)

i.e., multiplication of cardinal numbers is commutative.

(iii) $p \,.\, (q + r) = p \,.\, q + p \,.\, r$,

i.e., multiplication is distributive over addition.

(iv) $p \,.\, (q \,.\, r) = (p \,.\, q) \,.\, r$

i.e., associative law for multiplication holds.

(v) $p + (q + r) = (p + q) + r$

i.e., associative law for addition holds.

PROOF. (i) Let $P \cap Q = \phi$. The elements of an arbitrary set may be in any order, therefore

$P \cup Q = Q \cup P \quad \Rightarrow \quad card\,(P \cup Q) = card\,(Q \cup P)$

$\Rightarrow \quad card\ P + card\ Q = card\ Q + card\ P$.

(ii) We know that $P \times Q \sim Q \times P$ (See Ex. 2 above)

$\Rightarrow \quad card\,(P \times Q) = card\,(Q \times P)$

$\Rightarrow \quad p \,.\, q = q \,.\, p.$

(iii) Let Q, R be disjoint sets. Then

$$p.(q + r) = card\,[P \times (Q \cup R)] = card\,[(P \times Q) \cup (P \times R)]$$

$$= card\,(P \times Q) + card\,(P \times R) = p \,.\, q + p \,.\, r.$$

(iv) $\because \ P \times (Q \times R) \sim (P \times Q) \times R$ under the map $f(x,(y,z)) = ((x,y),z)$,

Hence $card\,[P \times (Q \times R)] = card\,[(P \times Q) \times R]$

or $\quad p \,.\, (q \,.\, r) = (p \,.\, q) \,.\, r.$

(v) Suppose that P, Q, R are pairwise disjoint sets. Also we know that

$(P \cup Q) \cup R = P \cup (Q \cup R)$

$\Rightarrow \quad card\,[(P \cup Q) \cup R] = card\,[P \cup (Q \cup R)]$

$\Rightarrow \quad (p + q) + r = p + (q + r)$

THEOREM 4. ***(i) If A_i is enumerable set for $i = 1, 2, 3, \ldots, n$ then $\bigcup_{i=1}^{n} A_i$ is enumerable and hence deduce that $n.a = a$.*** (GURUKUL KANGRI 2001; GARHWAL 1995)

(ii) If A_i is enumerable set for $i = 1, 2, 3, \ldots, n$, then $\bigcup_{i=1}^{n} A_i$ is enumerable and hence deduce that $a + a + a + \ldots$ to a terms $= a$

PROOF. (i) Let $A = \bigcup_{i=1}^{n} A_i$. We know that countable union of enumerable set is enumerable, hence A is enumerable.

DEDUCTION. Let $A_i \cap A_j = \phi$ for $i \neq j$; then by the definition of sum of cardinal numbers,

$$card\ A_1 + card\ A_2 + ... + card\ A_n = card\ A \quad \left(since \bigcup_{i=1}^{n} A_i = A\right) \qquad ...(1)$$

We know that cardinal number of an enumerable set is $\boldsymbol{\alpha}$ and since each $A_i (i = 1,2,3,...,n)$ is enumerable, so we have

$$card\ A_1 = card\ A_2 = ...card\ A_n = \boldsymbol{\alpha}$$

Also A is enumerable so card $A = \boldsymbol{\alpha}$. So from (1), $n \cdot \boldsymbol{\alpha} = \boldsymbol{\alpha}$

(ii) Let $A = \bigcup_{i=1}^{\infty} A_i$; then A is enumerable being enumerable union of enumerable sets and hence card $A = \boldsymbol{\alpha}$

DEDUCTION. Let $A_i \cap A_j = \phi$ for $i \neq j$; then card $A = \boldsymbol{\alpha}$ gives

$$\sum_{i=1}^{\infty} card\ A_i = \boldsymbol{\alpha}$$

$$\Rightarrow \quad \boldsymbol{\alpha} + \boldsymbol{\alpha} + \boldsymbol{\alpha} + \boldsymbol{\alpha} + ... \text{ to } \boldsymbol{\alpha} \text{ terms} = \boldsymbol{\alpha}.$$

THEOREM 5. ***(i) If A_i is non-enumerable set for $1 \leq i \leq n$, then $\bigcup_{i=1}^{n} A_i$ is non-enumerable and hence deduce that***

$$\boldsymbol{c + c + c + ... \text{ to } n \text{ terms} = c}.$$

(ii) If A_i is non-enumerable, $\forall i \in N$, then $\bigcup_{i=1}^{\infty} A_i$ is non-enumerable and hence deduce that

$$\boldsymbol{c + c + c + ... \text{ to } \alpha \text{ terms} = c.}$$

(GARHWAL 1991)

PROOF. (i) Each A_i is non-enumerable for $1 \leq i \leq n$, therefore card $A_i = \boldsymbol{c}$ for $1 \leq i \leq n$.

This imply $A_i \sim [a_i, a_{i+1})$ where $a_i, a_{i+1} \in \boldsymbol{R}$ for $i = 1,2,3,...,n$.

Let $a_i < a_{i+1}$ for $i = 1,2,3,...,n$. Thus, we have

$$A_1 \sim \{a_1, a_2\},$$
$$A_2 \sim \{a_2, a_3\},$$
$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots$$
$$A_n \sim [a_n, a_{n+1}).$$

At first suppose that $A_i \cap A_j = \phi$ for $i \neq j$; then $\bigcup_{i=1}^{n} A_i$ is equivalent to some subset of $[a_1, a_{n+1})$.

$$\therefore \quad card\left(\bigcup_{i=1}^{n} A_i\right) \leq \boldsymbol{c}. \qquad ...(1)$$

Now evidently

$$A_i \subseteq \bigcup_{i=1}^{n} A_i \quad \Rightarrow \quad card\ A_i \leq card\left(\bigcup_{i=1}^{n} A_i\right)$$

$\Rightarrow \quad c \le card\left(\bigcup_{i=1}^{n} A_i\right)$. ...(2)

Combining (1) and (2), we have

$$c \le card\left(\bigcup_{i=1}^{n} A_i\right) \le \mathbf{c}, \text{ i.e., } card\left(\bigcup_{i=1}^{n} A_i\right) = \mathbf{c}.$$

$\therefore \quad \bigcup_{i=1}^{n} A_i$ is non-enumerable.

Now suppose that $A_i \cap A_j = \phi$ for $i \neq j$, then

$$\bigcup_{i=1}^{n} A_i \sim [a_1, a_{n+1}) \Rightarrow card\left(\bigcup_{i=1}^{n} A_i\right) = card[a_1, a_{n+1})$$

$$= card\left(\bigcup_{i=1}^{n} A_i\right) = \mathbf{c}.$$

DEDUCTION. If we assume that $A_i \cap A_j = \phi$ for $i \neq j$, then as proved above, we have

$$\sum_{i=1}^{n} card\,(A_i) = \mathbf{c} \quad \left(\because card \bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} card\, A_i \text{ when } A_i \cap A_j = \phi \text{ for } i \neq j\right)$$

i.e., $\mathbf{c} + \mathbf{c} + \mathbf{c} + \ldots$ to $\boldsymbol{n}$ terms $= \mathbf{c}$.

(ii) Let $A = \bigcup_{i=1}^{\infty} A_i$, where card $A_i = c, \quad \forall i \in N$. Now

card $A_i = c \Rightarrow A_i \sim \left[1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}\right)$.

Thus, we have

$$A_1 \sim \left[0, \frac{1}{2}\right), A_2 \sim \left[\frac{1}{2}, \frac{3}{4}\right), A_3 \sim \left[\frac{3}{4}, \frac{7}{8}\right), \ldots \quad \ldots A_i \sim \left[1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}\right) \ldots .$$

Now assume that $A_i \cap A_j = \phi$ for $i \neq j$.

Then obviously $\bigcup_{i=1}^{\infty} A_i \sim [0,1)$.

$\therefore \quad card\left(\bigcup_{i=1}^{\infty} A_i\right) = card\ [0,1)$ or card $A = c$.

Next suppose that $A_i \cap A_j \neq \phi$ for $i = j$, then $\bigcup_{i=1}^{\infty} A_i$ is cardinally equivalent to some subset of $[0,1)$.

So, $\quad card\left(\bigcup_{i=1}^{\infty} A_i\right) \le \mathbf{c}$. ...(3)

Again $\quad A_i \subset \bigcup_{i=1}^{\infty} A_i \quad \Rightarrow \quad card\ A_i \le card\left(\bigcup_{i=1}^{n} A_i\right)$

$$\Rightarrow \mathbf{c} \le card\left(\bigcup_{i=1}^{\infty} A_i\right) \quad \ldots(4)$$

Combining (3) and (4) we get card $\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbf{c}$.

Hence $\bigcup_{i=1}^{\infty} A_i$ is non-enumerable.

DEDUCTION. Suppose that $A_i \cap A_j = \phi$ for $i \neq j$, we have

$$card\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathfrak{c} \quad \Rightarrow \sum_{i=1}^{\infty} card\, A_i = \mathfrak{c}$$

or $\mathfrak{c} + \mathfrak{c} + \mathfrak{c} + \ldots$ to $\mathfrak{a}$ terms $= \mathfrak{c}$.

THEOREM 6. ***Prove $\mathfrak{a} + \alpha = \alpha, \alpha$ being any transfinite cardinal number.***

(GURUKUL KANGRI 2001)

PROOF. If an enumerable set is added to an infinite set. Since A is an infinite set and therefore $\exists$ a subset B of A s.t. B is enumerable.

$\therefore$ card $B = \mathfrak{a}$.

Now we can write $A = (A - B) \cup B$.

$\therefore$ $A \cup N = (A - B) \cup B \cup N = (A - B) \cup (B \cup N)$.

B and N are enumerable sets $\Rightarrow B \cup N$ is enumerable

$\Rightarrow$ $B \cup N \sim N$.

Now $B \cup N \sim N, N \sim B \Rightarrow B \cup N \sim B$.

$\therefore$ $(A - B) \cup (B \cup N) \sim (A - B) \cup B$ i.e., $A \cup N \sim A$

or $card(A \cup N) = card\, A$ *i.e.*, $\alpha + \mathfrak{a} = \alpha$.

THEOREM 7. ***Show that $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$.*** (BANARAS 1977)

PROOF. Let $A = \{x : 0 \leq x \leq 1\}$. Then card $A = \mathfrak{c}$.

Now, let $B = \{(0, x) : x \in A\}$

Then obviously $B \subset A \times A$ $\therefore card B \leq card(A \times A)$

Again $A \sim B$ under the map f *s.t.*

$f(x) = (0, x)$, $\forall x \in A$, $\therefore card\, A \leq card\, B$.

$\therefore$ $card\, A \leq card\, (A \times A)$. ...(1)

Let x, y be any two real numbers in the closed interval [0, 1]. Then x and y can uniquely be expanded in the form of infinite decimals which contain non-zero digits.

Now define a map $g : (A \times A) \to A$ by writing

$$g(x, y) = 0.\ x_1 y_1 x_2 y_2 x_3 y_3 \ldots .$$

Obviously g is one-one. So by definition

$card\, (A \times A) \leq card\, A$. ...(2)

From (1) and (2), we get

$card\, A \leq card\, (A \times A) \leq card\, A$; $card\, (A \times A) = card\, A$

or $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$

1.13 TRANSCENDENTAL NUMBER

Definition. *A real number which is not an algebraic number is called Transcendental number. Thus the numbers e and π which are real but not algebraic numbers, are transcendental numbers.*

(MEERUT 2003)

All rational numbers are algebraic number hence every rational number is not transcendental, implying that every transcendently number must be irrational, for

$\boldsymbol{R}$ = (rational numbers) $\cup$ (irrational numbers).

It must be noted by the readers that there are so many irrational numbers which are algebraic e.g., $(n)^{1/r}$. Therefore every irrational number is not transcendental number.

THEOREM 1. ***Prove that every monotonic function in a closed interval is discontinuous at a countable number of points of that interval.***

PROOF. Let $f(x)$ be a monotonic function in the closed interval $[a, b]$. Also let it be a monotonically increasing function and be discontinuous at an arbitrary point x. Then

$$\delta(x) = f(x+0) - f(x-0) > 0 \qquad \ldots(1)$$

where $f(a) = f(a-0), f(b) = f(b+0)$.

Let $\xi_1, \xi_2, \ldots, \xi_{m-1}$ be numbers in the intervals $x_k < \xi_k < x_{k+1}$ where $a < x_1 < x_2 < \ldots x_m < b$, where $\xi_0 = a$ and $\xi_m = b$.

$$\therefore \quad f(\xi_k) - f(\xi_{k-1}) \geq f(x_k + 0) - f(x_k - 0) = \delta(x_k) \qquad \text{[by (1)]} \qquad \ldots(2)$$

Therefore $f(b) - f(a) = \sum_{k=1}^{m} [f(\xi_k) - f(\xi_{k-1})] \geq \sum_{k=1}^{m} \delta(x_k)$.

Let $\delta(x_k) > \frac{1}{n}, \ \forall k$.

Then by last inequality, we have

$$f(b) - f(a) > \frac{m}{n} \quad \text{or} \quad [f(b) - f(a)]n > m.$$

This shows that m which is the number of points of discontinuity x with $\delta(x) > \frac{1}{n}$ is bounded above, *i.e.*, the number of points of discontinuity x with $\delta(x) > \frac{1}{n}$ are finite in the closed interval $[a, b]$. Since $n \in \boldsymbol{N}$, we see that the number of points of discontinuity x with $\delta(x) > \frac{1}{n}$ are finite in the closed interval $[a, b]$. Since every finite set is countable and for every $x \ \exists \, n \in \boldsymbol{N}$ therefore the number of points of discontinuity in the closed interval $[a, b]$ will be an enumerable union of countable sets and hence countable. Hence the theorem.

THEOREM 2. **(Cantor's Theorem).** ***Prove that card A < card P(A), P(A) being power set of the set A.*** (MEERUT 1983,91; GURUKUL KANGRI 2001)

Or

Show that for every cardinal number n, $2^n > n$.

PROOF. Let $B^* = \{\{x\} : x \in A\}$; then obviously $B^* \subset P(A)$. Now define a map $f : A \to B^*$ *s.t.* $f(x) = \{x\}$. Obviously $A \sim B^*$. Hence

$$card\ A \leq card\ P(A).$$

Thus we are only to show that $card\ A \neq card\ P(A)$,

i.e., $A \propto P(A) \qquad \ldots(1)$

Suppose contradiction, *i.e.*, $A \sim P(A)$. So $\exists$ a one-one map $f : A \xrightarrow{onto} P(A)$.

Let $B = \{x \in A : x \notin f(x)\}$.

Clearly, $B \subset A \ \Rightarrow \ B \in P(A)$.

Since the mapping f is onto, there must exist $x \in A$ *s.t.* $f(x) = B$. Now if $x \in B$ then by definition of $B, x \in f(x)$ which is not possible.

Consider the second possibility that $x \notin B$, then $x \in f(x) = B$ which is again impossible.

It means that our assumption is wrong. Hence (1) is true.

DEDUCTION. (i) From the last theorem, it following that $n < 2^n$ where card $A = n$ and $cardP(A) = 2^{cardA} = 2^n$. (MEERUT 1988)

(ii) Also $2^a > a$.

THEOREM 3. **(Equivalence Theorem).** ***If*** $A_1 \subset B \subset A$ ***and*** $A \sim A_1$, ***then*** $A \sim B$.

Or

If $A_1 \subset B \subset A$ ***and card*** A = ***card*** A_1, ***then card*** A = ***card*** B.

PROOF. $A \sim A_1 \Rightarrow \exists$ a one-one map $f : A \xrightarrow{onto} A_1$.

As $B \subset A$ so f_B is a one-one onto where f_B is the restriction of f to B. This means that $B \sim B_1 \subset A_1$. Similarly, $A_1 \sim A_2 \subseteq B_1$.

Continuing in this way we get equivalent sets

$$A, A_1, A_2, \ldots \text{ and } B, B_1, B_2, \ldots$$

s.t. $$A \supset B \supset A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset B_3 \supset \ldots$$

Let $$S = A \cap B \cap A_1 \cap B_1 \cap A_2 \cap B_2 \cap \ldots$$

Then we can write

$$A = (A - B) \cup (B - A_1) \cup (A_1 - B_1) \cup \ldots \cup S,$$
$$B = (B - A) \cup (A_1 - B_1) \cup (B_1 - A_2) \cup \ldots \cup S$$

Define a map $\psi : A \to B$ *s.t*.

$$\psi(A - B) = A_1 - B_1,$$
$$\psi(A_1 - B_1) = A_2 - B_2,$$
$$\psi(A_2 - B_2) = A_3 - B_3,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\psi(B - A_1) = B - A_1,$$
$$\psi(B_1 - A_2) = B_1 - A_2,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\psi(S) = S.$$

Above definition of ψ makes the mapping ψ one-one and showing that $A \sim B$.

THEOREM 4. **(Schorder-Bernstein Theorem).** ***If card*** $A \leq$ ***card*** B ***and card*** $B \leq$ ***card*** A, ***then card*** A = ***card*** B. (MEERUT 1989)

Or

If each of the sets A ***and*** B ***is equivalent to a subset of other, then*** $A \sim B$.

PROOF. Let f and g be one-to-one mapping from A onto B and from B into A respectively. Let $f(A) = B_i \subset B$ and $g(B) = A_2$ and $g(B_1) = A_3$, then we have $A \supset A_2 \supset A_3$. Further $g(B_1) = A_3$, $f(A) = B_1$ implies $g(f(A)) = A_3$, giving gof is a one-to-one mapping from A to $A_3 \Rightarrow$ *card* A = *card* A_3. Hence by the above theorem, *card* A = *card* A_2. Also existence of g *s.t.* $g(B) = A_2$ shows that *card* B = *card* A_2. Hence *card* B = *card* A.

THEOREM 5. ***Show that*** $2^a = \mathbf{c}$.

PROOF. We know that *card* $[0,1] = \mathbf{c}$. On the other hand each $x \in [0,1]$ can be written in the form of binary expansion as $x \in 0.x_1x_2x_3\ldots$, where each $x_i = 0$ or 1.

But selecting each x_i in two ways (either 0 or 1) we can form at most 2^a numbers. So the *card* $[0,1] = 2^a$ implying that $2^a = \mathbf{c}$.

THEOREM 6. ***Every superset of an uncountable set is uncountable.***

PROOF. Suppose contradiction, *i.e.*, if B is the superset of an uncountable set A, then B is countable. But we know that every subset of a countable set is countable and so A must be countable which is a contradiction and so B is uncountable. That is to say that every superset of an uncountable set is uncountable.

THEOREM 7. ***Union of two enumerable sets is also enumerable.***

PROOF. Let A and B be the two enumerable sets.

CASE I. When $A \cap B = \phi$.

Let $A = \{a_1, a_2, ...\}, B = \{b_1, b_2, ...\}$. Now establish correspondence

$$f : A \cup B \to \boldsymbol{N}$$

s.t. $f(a_n) \to 2n-1$ (odd positive integer),

$f(b_n) \to 2n$ (even positive integer).

Evidently this mapping is one-one from $A \cup B$ onto $\boldsymbol{N}$.

CASE II. When $A \cap B \neq \phi$, then we can write $A \cup B = A \cup (B - A)$. Taking $B_1 = B - A$ we have $A \cap B_1 = \phi$. As already proved $A \cup B_1$ is countable where B_1 is countable where B_1 is countable being the subset of countable set B and hence $A \cup B = A \cup B_1$ is countable when B_1 is countable infinite but if B_1 is finite say

$$B_1 = \{e_1, e_2, ..., e_m\} .$$

Then $A \cup B = A \cup B_1 = \{e_1, e_2, e_3, ..., e_m, a_1, a_2, a_3, ...\}$.

Now set a correspondence

$$f : \boldsymbol{N} \to A \cup B_1$$

s.t. $f(i) = e_i, \quad 1 \le i \le m$,

$f(m+i) = a_i, \quad \forall i$

$\Rightarrow$ $A \cup B$ is enumerable.

We can generalize the result that union of two countable sets is countable (whether each is countably infinite or finite).

THEOREM 8. ***Every infinite set is equivalent to its proper subset.***

PROOF. **CASE I.** When A is countably infinite, then A can be written as a sequence. Let $A = \{a_1, a_2, a_3 ...\}$. Then the function $f(a_n) = a_{n+1}$ establishes a one to one correspondence between the set A and $A - \{a_1\}$ which is a proper subset of A.

CASE II. When A is uncountably infinite, then it has an enumerable subset say B where $B = \{a_1, a_2, a_3 ...\}$. We shall show that $A \sim (A - \{a_i\})$.

Write C = A – B; then $A = B \cup C$ and $B \cap C = \phi$. Also $A - \{a_i\} = B - \{a_i\}) \cup C$. Let $e(x)$ be the identity mapping which associates each $x \in A$ onto itself. Let f be the function: $f(a_n) = a_{n+1}$. Now define a function h:

$$h(x) = \begin{cases} e(x) : x \in C, \\ f(x) : x \in B. \end{cases}$$

Then the range of h is $(B - \{a_i\}) \cup C$ which is a proper subset of $B \cup C = A$. Thus the result follows.

THEOREM 9. ***If*** α ***and*** β ***are cardinal numbers such that*** $\alpha \le \beta$ ***and*** $\beta \le \alpha$ ***, then*** $\alpha = \beta$ **.**

PROOF. Let card $A = \alpha$, card $B = \beta$

$\alpha \le \beta \Rightarrow$ *card* $A \le$ *card* B

$\Rightarrow$ $A \sim B$ or $A \sim$ to a subset of B ...(1)

$\beta \le \alpha \Rightarrow B \sim A$ or $B \sim$ to a subset of A . ...(2)

(1) and (2) give the required result.

THEOREM 10. ***If an enumerable set is subtracted from an enumerable set, the remaining set will be enumerable.***

PROOF. Suppose contradiction, *i.e.*, $A-B$ is non-enumerable where A and B are enumerable sets. We can write $A=(A-B)\cup B$. Now $A-B$ is non-enumerable gives A to be non-enumerable which is a contradiction. Hence the result follows.

THEOREM 11. ***If we subtract an enumerable set from a non-enumerable set, then the remaining set is non-enumerable.***

PROOF. Suppose contradiction, *i.e.*, $A-B$ is enumerable where A is non-enumerable and B is an enumerable set. We can write $A=(A-B)\cup B$. Since both $A-B$ and B are enumerable sets, it implies A is enumerable which a contradiction as A is non-enumerable. Hence the result.

SOLVED EXAMPLES

EXAMPLE 1. ***Prove that the set Z of all integers is countable.***

(MEERUT 1993,98,2004; KANPUR 2001)

SOLUTION. Write

$$\boldsymbol{Z}^+=\{1,2,3,\ldots\},$$

$$\boldsymbol{Z}^-=\{-1,-2,-3,\ldots\}.$$

Then, we have $\boldsymbol{Z}^+\sim\boldsymbol{N}$ under the mapping $n\to n$.

Now $\boldsymbol{Z}^+\sim\boldsymbol{N}$ under the mapping $n\to n$,

$\boldsymbol{Z}^-\sim\boldsymbol{N}$ under the mapping $(-n)\to n$.

Also singleton set $\{0\}$ is finite, so countable. Thus $\boldsymbol{Z}$ is the countable union of countable sets and hence countable.

Alternatively, define the mapping $f:\boldsymbol{N}\to\boldsymbol{Z}$

$$s.t.\, f(x)=(-1)^x\left[\frac{x}{2}\right],$$

where $\left[\frac{x}{2}\right]$ represents the integral value of $\frac{x}{2}$, *i.e.*, represents the largest integer less than or equal to $\frac{x}{2}$.

Establish that this mapping gives a one-to one correspondence between $\boldsymbol{N}$ and $\boldsymbol{Z}$ implying that $\boldsymbol{Z}$ is countable.

Alternatively the $f:\boldsymbol{N}\to\boldsymbol{Z}$ *s.t.*

$$f(n)=\frac{n-1}{2} \text{ if } n \text{ is odd} \in \boldsymbol{N}$$

$$=-\frac{n}{2} \text{ if } n \text{ is even} \in \boldsymbol{N}.$$

This is also a one-to one correspondence $\Rightarrow \boldsymbol{Z}\sim\boldsymbol{N}$.

REMARK

It shows that an infinite set can be equivalent to its proper subset, *e.g.*, $\boldsymbol{Z}\sim\boldsymbol{N}$.

EXAMPLE 2. ***Find the power of an aggregate of numbers given by*** $\frac{M}{2^m}$, ***M and m being positive and integral.*** (GARHWAL 1994,98)

(Power of a set means the cardinal number of that set).

SOLUTION. Let us suppose that

$$B = \left\{\frac{M}{2^m} : M,\ m \in \boldsymbol{N}\right\}.$$

Write $B_M = \left\{\frac{M}{2^m} : m \in \boldsymbol{N}\right\}$.

Then we have,

$$B_1 = \left\{\frac{1}{2^1}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots\right\}$$

$$B_2 = \left\{\frac{2}{2^1}, \frac{2}{2^2}, \ldots, \frac{2}{2^n}, \ldots\right\}$$

$$B_3 = \left\{\frac{3}{2^1}, \frac{3}{2^2}, \ldots, \frac{3}{2^n}, \ldots\right\}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$B_n = \left\{\frac{n}{2^1}, \frac{n}{2^2}, \ldots, \frac{n}{2^n}, \ldots\right\}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

Evidently,

(i) B_i is enumerable $\forall i \in \boldsymbol{N}$ under the mapping $\frac{i}{2^n} \to n$,

(ii) $B_i's$ are pairwise disjoint.

(iii) $B = \bigcup_{i=1}^{\infty} B_i$.

Thus B is enumerable being the enumerable union of enumerable sets. Hence card $B = a$, *i.e.*, the power of the given set is a.

EXAMPLE 3. ***Prove that*** $\boldsymbol{a < c}$.

SOLUTION. Since $\boldsymbol{N \subset R} \Rightarrow card\ \mathrm{N} < card\ R$

$$\Rightarrow \quad \boldsymbol{a < c}.$$

EXAMPLE 4. ***Prove that*** $\boldsymbol{card P(A) = 2^{cardA}}$ ***for any finite set A.*** (MEERUT 1988)

SOLUTION. Let $card\ A = \alpha$.

Then $card\ (P(A)) = 1 + {}^{\alpha}c_1 + {}^{\alpha}c_2 + \ldots + {}^{\alpha}c_{\alpha} = (1+1)^{\alpha} = 2^{\alpha} = 2^{\alpha} = 2^{cardA}$.

EXAMPLE 5. ***Prove that*** $\alpha \le \alpha$ ***for any cardinal number*** α.

SOLUTION. Let card $P = \alpha$.

Define an identity map $f : P \to P$ written by

$$f(x) = x, \forall x \in P.$$

Obviously, f is one-one. Hence by definition,

$$card\ P \le card\ P \quad i.e., \quad \alpha \le \alpha.$$

EXAMPLE 6. ***If each $a_i (i = 1, 2, ..., n)$ is a rational number, then the point $a = (a_1, a_2, ..., a_n) \in R^n$ is called a rational point. Show that the set of all rational points in R^n is denumerable.***

SOLUTION. We know that set of rational numbers is countable. So varying $a_1, a_2, ..., a_n$, we can form a^n rational points.

But we know that $a \cdot a = a$ and hence $a^n = a$.

$\Rightarrow$ set of all rational points has the same cardinal number as that of $\boldsymbol{N}$.

$\Rightarrow$ set of rational points is countable.

EXAMPLE 7. ***Prove that $c^c = 2^c$.***

SOLUTION. We have $c^c = (2^a)^c = 2^{ac} = 2^c$.

EXAMPLE 8. ***Let X be any non-empty set and let C be the family of functions $f : x \to \{0, 1\}$. Then show that the family of subset of X i.e., the power set of X is equivalent to C.***

SOLUTION. Let $A \in P(X)$ where $P(X)$ denotes the power set of X. Also let ϕ_A denote the characteristic function of A relative to X.

Now define a map $f : P(X) \to C$ by the formula $f(A) = \phi_A$.

Obviously f as defined above is one-one onto. Hence, $P(X) \sim C$.

EXAMPLE 9. ***Prove that $]0, 1] \sim]0, 1[$.***

SOLUTION. Denote the points of]0, 1] by x and of]0, 1[by y. Now define a correspondence

$$y = \frac{3}{2} - x \text{ for } \frac{1}{2} < x \le 1 \text{ ; then } \frac{1}{2} \le y < 1,$$

$$y = \frac{3}{4} - x \text{ for } \frac{1}{4} < x \le \frac{1}{2} \text{ ; then } \frac{1}{4} \le y < \frac{1}{2},$$

$$y = \frac{3}{8} - x \text{ for } \frac{1}{8} < x \le \frac{1}{4} \text{ ; then } \frac{1}{8} \le y < \frac{1}{4},$$

and so on.

From the above correspondence, we see that for every $x \in]0,1]$, there corresponds one and only one y of $]0,1[$. Hence by definition $]0,1] \sim]0,1[$.

EXAMPLE 10. ***Show that for every real number x, the real number in the semi-open interval $[x, x + 1)$ form an uncountable set.***

SOLUTION. Let x be any real number. Define a function

$$f : [x, x+1) \to [0,1) \text{ given by } f(y) = y - x.$$

Then f is well defined, for obviously,

$$f(x) = x - x = 0,\ f(x+1) = x + 1 - x = 1.$$

Again $f(y_1) = f(y_2) \Rightarrow y_1 - x = y_2 - x \Rightarrow y_1 = y_2 \Rightarrow f$ is one-one.

Also f is a continuous map which implies that f is an onto map.

Hence $[x, x+1) \sim [0,1)$

$\therefore$ $card\ [x, x+1) = card\ [0,1)$.

Now we know that the set of all real numbers in the set semi-open interval [0, 1) is uncountable and hence the set of all real numbers in $[x, x + 1)$ which is cardinally equivalent to [0, 1) is uncountable.

EXAMPLE 11. ***If α is any transfinite cardinal number, then $\mathbf{a} \leq \alpha$.***

SOLUTION. Let A be an infinite arbitrary set *s.t.* card $A = \alpha$.

Now, A is infinite set $\Rightarrow \exists$ an enumerable subset B of $A \Rightarrow card\ B = \mathbf{a}$

$$B \subseteq A \quad \Rightarrow \quad card\ B \leq card\ A \quad \Rightarrow \quad \mathbf{a} \leq \alpha .$$

EXAMPLE 12. ***Show that the set of all transcendental numbers in any interval is non-enumerable.*** (MEERUT 1995, 2003; PUNJAB UNIV. 2002)

SOLUTION. We know that the set of all algebraic numbers and transcendental numbers is the set of all real numbers which is known to be uncountable. Also we know that the set of algebraic numbers in an interval is enumerable.

But we have already proved that if an enumerable set is removed from a non-enumerable set, the remaining set is non-enumerable. Therefore the complement of the set of all algebraic numbers in any interval relative to the set of all real numbers in that interval is uncountable. But this is the set of all transcendental numbers. Hence the result.

EXAMPLE 13. ***Show that the interval (0, 1) is equivalent to the set R of all real numbers and hence show that card (0,1) = card R.***

SOLUTION. Define a function $f : (0,1) \to \boldsymbol{R}$ *s.t.*

$$f(x) = \begin{cases} \dfrac{2x-1}{x}, x \in \left(0, \dfrac{1}{2}\right) \\ \dfrac{2x-1}{1-x}, x \in \left[\dfrac{1}{2}, 1\right) \end{cases}.$$

which show that this function is one-one and onto implying that $(0,1) \sim \boldsymbol{R}$ and hence $card\ (0,1) = card\ R = \mathbf{c}$.

Also since $(0,1)$ is uncountable, the set $\boldsymbol{R}$ is also uncountable.

Above property supports our idea of defining the same cardinal numbers $\mathbf{c}$ of the two sets $(0,1)$ and $\boldsymbol{R}$. Then $\mathbf{c}$ is called the Cardinal number of continuum.

EXAMPLE 14. ***Prove that $\mathbf{a} < \mathbf{c} < \mathbf{f}$ where $\mathbf{a}, \mathbf{c}$ and $\mathbf{f}$ denote the cardinal numbers of set of all natural numbers, and real numbers and set of all real valued functions defined over [0,1] respectively.*** (GARHWAL 1997; MEERUT 1983,89)

SOLUTION. We have already proved that $\mathbf{a} < \mathbf{c}$. Now it remains to prove that $c < f$. Let F be set of all real valued functions defined over [0, 1].

Now consider the mapping $f_k : [0,1] \to \boldsymbol{R}$ defined as $f_k(x) = k, \quad \forall x \in [0,1]$ and k being a real number in [0, 1].

All these functions are real valued and so that set $F^* = \{f_k : 0 \leq k_1\}$ is a proper subset of the set F.

We can set up a one-to-one correspondence between $[0,1]$ and F^* $(s.t.\ F^* \subset F)$. Hence $card\ [0,1] = card\ F^* < card\ F$ or $\mathbf{c} < \boldsymbol{f}$.

EXAMPLE 15. ***Show that a countable set is a Borel set.***

SOLUTION. Let $A = \{a_1, a_2, a_3, ...\}$ be a countable set. Now note that

$$\{x : x = a_r\} = \bigcap_{n=1}^{\infty} \left\{x : a_r \le a_r + \frac{1}{n}\right\}$$

and $$A = \bigcup_{r \in N} \{a_r\}$$

$\Rightarrow$ A is obtained by the formation of countable union and intersection of closed and open sets and hence A is a Borel set.

1.14 CONTINUUM HYPOTHESIS

It is assumed that there is no cardinal number between **a** and **c**. Thus **c** is assumed to be the second transfinite cardinal number. However, there are other cardinal number greater than **c**. For instance *card* $P(R) > \mathbf{c}$.

REVIEW QUESTIONS AND ARCHIVE

1. (a) Define an enumerable set. Show that the set of real numbers can not be enumerable, although the set of rationales is enumerable.

(b) If $f : A \to B$ and the range of f is uncountable, prove that domain of f is also uncountable.

2. Define cardinal number of a set. Show that $n < 2^n$ for any cardinal number n.

3. Prove that the set of all real numbers in the closed interval [0, 1] is uncountable.

4. Prove that if A and B are enumerable then $A \times B$ is also enumerable.

5. Prove that $\alpha + \alpha = \alpha$ for any infinite cardinal number α.

6. Find the cardinal number of the set $\{x\}$ of those numbers in the interval [0,1] whose ternary expansion does not have the digit 1.

7. Prove the following:

(i) $[0,1] \sim]0,1[$,

(ii) $[0,1] \sim [2,5]$,

(iii) $[0,1[\sim]0,1[$.

8. If $\{E_n\}$ be a sequence of countable sets and $S = \bigcup_{n=1}^{\infty} E_n$, then prove that S is countable. (KANPUR 2003)

9. Let α and β be any two cardinal numbers such that $\alpha \le \beta$ and $\beta \le \alpha$, then prove that $\alpha = \beta$.

10. (i) Prove that the set of all numbers in any interval can not be enumerable.

(ii) Show that the set of all characteristic functions on $\boldsymbol{R}$ is uncountable.

11. Set of real numbers is ..... (KANPUR 2003)

12. Every isolated set of point is ..... (KANPUR 2002)

13. By an example show that cancellation law does not hold in case of cardinal numbers?

14. Set of integers $\{0,1,2,...\}$ is uncountable. (KANPUR 2003)

15. Show that the family of all finite subsets of the natural numbers is countable infinite.

16. Prove that

(i) $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{c}$ (ii) $\boldsymbol{a} + \boldsymbol{a} = \boldsymbol{a}$

(iii) $\boldsymbol{c} + \boldsymbol{c} = \boldsymbol{c}$ (iv) $\boldsymbol{a} \cdot \boldsymbol{c} = \boldsymbol{c}$

(v) $\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{a}$

17. State and prove Schroder-Bernsteing theorem.

18. Prove that the set of complex numbers is uncountable.

19. Exhibit $a\ 1-1$ correspondence between the points of the closed interval $[0,1]$ of $\boldsymbol{R}$ and the points of the half closed interval $(0,1]$ of $\boldsymbol{R}$.

20. Show that the set of all polynomial functions with integer (rational) coefficients is countable (or say has the cardinal $\boldsymbol{\alpha}$)

21. Prove that card $P(A) = 2^{card\ A}$, where A is any finite set. (GARHWAL 1992)

22. Using the mapping $f : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$ given by $f(x,y) = 2^x(x^y + 1) - 1$, show that set $\boldsymbol{N} \times \boldsymbol{N}$ is countable equivalent.

23. Show that the set of points in the closed interval $[2,4]$ and in the open interval $(1,2)$ are cardinally equivalent. (MEERUT 2004)

24. If a finite set of elements in added to an enumerable set, the executing set is also enumerable.

25. If B is a countable subset of an uncountable set A. Then $A - B$ is ..... (MEERUT 2001)

26. Show that the set of all sequence whose elements are the digit 0 and 1 is uncountable. (KANPUR 2001; PUNJAB 2003)

27. If $\{E_n\}$ be a sequence of countable sets and $S = \bigcup_{n=1}^{\infty} E_n$ then prove that S is countable.

CHAPTER SUMMARY

This chapter introduced the fundamental notions and basic concepts. The important points discussed in this chapter are as follows :

- The numbers 1, 2, 3,... are called natural numbers. We represent the set of natural numbers by $\boldsymbol{N}$, *i.e.*, $\boldsymbol{N} = \{1, 2, 3, \ldots\}$
- The numbers ... ..., –3, –2, –1, 0, 1, 2, 3, ... are called integers. We represent the set of integers by $\boldsymbol{Z}$, *i.e.*, $\boldsymbol{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Any number of the form p/q, where $p, q \in \boldsymbol{Z}$, $q \neq 0$ and p, q have no common factor (except ± 1) is called a rational number.
- A number which is either rational or irrational is called a real number. The set of real numbers is denoted by $\boldsymbol{R}$.
- A set is a well defined collection of objects
- A set containing no elements is called empty set and is denoted by the symbol ϕ.
- Set containing only one element is a Singleton set. The set {a} is a singleton set.
- The supremum of a set $S \subset \boldsymbol{R}$, if exists, is unique.
- The infimum of a set, if exists, is unique.
- A sequence is represented as $< s_n >$ or $\{s_n\}$, when s_n is the n^{th} term of the sequence.
- The set of all distinct terms of a sequence is called the range set of that sequence.
- A sequence, whose range is a subset of $\boldsymbol{R}$ is called a real sequence or a sequence of real numbers.

FOR ADVANCED LEARNERS

- If $x > 0$, $y > 0$ then $xy > 0$; if $x > 0$, $y > 0$ then $0 > xy$. If $0 > x$, $0 > y$ then $xy > 0$ and these results hold with $>$ replaced everywhere by $\geq$.
- If S is a non-empty set of integers then $m = \sup S$ is an integer.
- There exists $n \in \boldsymbol{Z}$ such that $n - 1 \leq x < n$.
- If $x > 0$, $y \geq 0$ then there exists $n \in \boldsymbol{N}$ such that $nx > y$.
- If $q \in \boldsymbol{Q}$ such that $x < q < y$. If we choose integers $n > \frac{1}{y-x}$ and $k \geq ny$ and let m be the least integer such that $y \leq \frac{m}{n}$. Then $x < \frac{(m-1)}{n} < y$
- If S and T are non-empty set of positive numbers then
  $\sup\{st : x \in S, t \in T\} = \sup S \times \sup T$
- Following conditions are equivalent on non-empty subset X and Y of $\boldsymbol{R}$
  (i) $x \leq y \ \forall \ x \in X, y \in Y$
  (ii) $\exists \ r \in \boldsymbol{R}$ such that $x \leq r < y \ \forall \ x \in X$ and $y \in Y$
- If $0 < a \neq 1$ and $a^x = 1$ then $x = 0$
- If $a > 0$ and $x > y$. If $a > 1$ then $a^x > a^y$ and that if $a < 1$ then $a^x < a^y$.
- If $a > 0$ then for each $x > 0$ there exists a unique $y \in \boldsymbol{R}$ such that $a^y = x$.
- If $a > 1$ then the function $\log_a x$ is strictly increasing and that if $0 < a < 1$ then $\log_a x$ is strictly decreasing.
- If in the system of axioms for $\boldsymbol{R}$, the least upper bound principle is replaced by the axioms of Acrhimedes, then the nested interval principle is equivalent to the completeness of $\boldsymbol{R}$.