CHAPTER 1

PRELIMINARIES

1.1 Introduction

The concept of sets, functions and basic properties of the real number system such as bounded sets, intervals and bounded functions are used throughout this book. In this chapter we present a brief summary of the concepts and notations in the study of sets, functions, intervals, bounded sets, the greatest lower bound and the least upper bound.

1.2 Sets and functions

The concepts of sets and functions are indispensable to almost all branches of pure mathematics. The usual materials of elementary set theory is so current that we take it for granted. We freely use the following notions of set theory.

(i) A is a subset of B written as $A \subseteq B$.

(ii) Union of two sets A and B written as $A \cup B$.

(iii) Intersection of two sets A and B written as $A \cap B$.

(iv) Complement of a subset A of X written as A^c.

(v) Difference of two sets A and B written as $A - B$.

(vi) Cartesian product of two sets A and B written as $A \times B$.

(vii) A function f from a set A to a set B written as $f : A \to B$.

(viii) The empty set $\emptyset$ which contains no element.

Certain letters are reserved to denote particular sets which occur often in our discussion. They are

$\mathbf{N} = \{1, 2, 3, \ldots, n, \ldots\}$ the set of all *natural numbers.*

$\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots, \pm n, \ldots\}$ the set of all *integers.*

$\mathbf{Q} \ldots$ the set of all *rational numbers.*

$\mathbf{Q}^+ \ldots$ the set of all *positive rational numbers.*

$\mathbf{R} \ldots$ the set of all *real numbers.*

$\mathbf{R^n} \ldots$ the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers.

$\mathbf{C} \ldots$ the set of all *complex numbers.*

$\mathbf{C^n} \ldots$ the set of all ordered n-tuples $(z_1.z_2, \ldots, z_n)$ of complex numbers.

1.3 Intervals in R

We use the order structure in the real number system $\mathbf{R}$ to define certain subsets of $\mathbf{R}$ called intervals.

Let $a, b \in \mathbf{R}$ and $a < b$.

(i) $(a, b) = \{x | x \in \mathbf{R}, a < x < b\}$ is called the *open interval* with a and b as end points.

(ii) $[a, b] = \{x | x \in \mathbf{R}, a \leq x \leq b\}$ is called the *closed interval* with a and b as end points.

(iii) $(a, b] = \{x | x \in \mathbf{R}, a < x \leq b\}$ is called the *open-closed interval* with a and b as end points.

(iv) $[a, b) = \{x | x \in \mathbf{R}, a \leq x < b\}$ is called the *closed-open interval* with a and b as end points.

(v) $[a, \infty) = \{x | x \in \mathbf{R}$ and $x \geq a\}$.

(vi) $(a, \infty) = \{x | x \in \mathbf{R}$ and $x > a\}$.

(vii) $(-\infty, a] = \{x | x \in \mathbf{R}$ and $x \leq a\}$.

(viii) $(-\infty, a) = \{x | x \in \mathbf{R}$ and $x < a\}$.

(ix) $(-\infty, \infty) = \mathbf{R}$.

Any subset of R which is one of the above forms is called an *interval*. Any interval of form (i), (ii), (iii) or (iv) is called a *finite interval* or *bounded interval* and any interval of the form (v), (vi), (vii) (viii) or (ix) is called an *infinite interval* or an *unbounded interval*.

1.4 Bounded sets

Definition. A subset A of $\mathbf{R}$ is said to be *bounded above* if there exists an element $\alpha \in \mathbf{R}$ such that $a \leq \alpha$ for all $a \in A$. The real number α is called an *upper bound* of A.

A is said to be *bounded below* if there exists an element $\beta \in \mathbf{R}$ such that $a \geq \beta$ for all $a \in A$. The real number β is called a *lower bound* of A.

A is said to be *bounded* if it is both bounded above and bounded below.

Note. 1. Let $A \subseteq R$. If $\alpha \in \mathbf{R}$ is an upper bound of A. Any real number x with $x > \alpha$ is also an upper bound of A. Thus a set which is bounded above has infinite number of upper bounds. Similarly a set which is bounded below has infinite number of lower bounds.

2. Let $A \subseteq \mathbf{R}$ and $x \in \mathbf{R}$. Then x is not an upper bound of A if and only if there exists at least one element $a \in A$ such that $a > x$. Similarly x is not a lower bound of A if and only if there exists at least one element $a \in A$ such that $a < x$.

Examples. 1. Let $A = \{2, 3, 5\}$. Any element $x \in \mathbf{R}$ such that $x \leq 2$ is a lower bound of A and any element $x \in \mathbf{R}$ such that $x \geq 5$ is an upper bound of A.

2. Let $A = \mathbf{N}$. Any real number x is not an upper bound of $\mathbf{N}$, since there exists a natural number n such that $n > x$. Hence $\mathbf{N}$ is not bounded above. However $\mathbf{N}$ is bounded below. Any real number $x \leq 1$ is a lower bound of $\mathbf{N}$.

3. Let $A = \mathbf{Z}$. Then $\mathbf{Z}$ is neither bounded above nor bounded below.

4. Let $A = \{x|x \leq 2\} = (-\infty, 2]$. The set A is bounded above but not bounded below. Any real number $x \geq 2$ is an upper bound of A.

5. Let $A = (0,1) = \{x|x \in \mathbf{R} \text{ and } 0 < x < 1\}$. Here any number $y \geq 1$ is an upper bound of A. Hence $[1, \infty)$ is the set of all upper bounds of A. We notice that the least upper bound of A is 1.

1.5 Least upper bound and greatest lower bound

Definition. Let $A \subseteq \mathbf{R}$ and $u \in \mathbf{R}$. The real number u is called the *least upper bound* (*lub*) or *supremum* (*sup*) of A if

(i) u is an upper bound of A

(ii) If $v < u$, then v is not an upper bound of A.

Let $A \subseteq \mathbf{R}$ and $\ell \in \mathbf{R}$. The real number ℓ is called the *greatest lower bound* (*g.l.b*) or *infimum* (*inf*) of A if

(i) ℓ is a lower bound of A

(ii) If $m > \ell$, then m is not a lower bound of A.

Examples. 1. Let $A = \{1, 3, 5, 6\}$.

glb of $A = 1$ and *lub* of $A = 6$. In this case both *glb* and *lub* belongs to A.

2. Let $A = (0, 1)$.

glb of $A = 0$ and *lub* of $A = 1$. In this case both *lub* and *glb* do not belong to A.

3. Let $A = [a, b)$.

glb of $A = a$ and *lub* of $A = b$. Here $glb \in A$ and $lub \notin A$.

4. Let $A = \mathbf{N}$.

$\mathbf{N}$ is not bounded above and hence $\mathbf{N}$ does not have any *lub*. However *glb* of $\mathbf{N} = 1$.

Exercises

1. Find the *lub* and *glb* of each of the following sets if they exist. State whether *lub* and *glb* belong to the set or not.

   (i) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$

(ii) $A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$

(iii) $A = [-3, 1)$ (iv) $A = \{1\}$

(v) $A = (0, 100)$ (vi) $A = (-\infty, 1)$

(vii) $A = \{x | x^2 < 2\}$ (viii) $A = (-1, 3) \cup (5, 6)$

(ix) $A = \left\{\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \dots, \pm\frac{1}{n}, \dots\right\}$

(x) $A = \{1, -2, 3, -4, 5, \dots\}$

(ix) $A = Q$ (xii) $A = 2\mathbf{N} = \{2, 4, 6, \dots, 2n, \dots\}$.

2. Show that the *lub* and the *glb* of a set if they exist are unique.

1.6 Bounded functions

Definition. Let $f : A \to \mathbf{R}$ be any function. Then $\{f(a) | a \in A\}$ is called the *range of* f, which is a subset of $\mathbf{R}$. The function f is said to be a *bounded function* if its range

is a bounded subset of $\mathbf{R}$. Hence f is a bounded function if and only if there exists a real number m such that $|f(x)| \leq m$ for all $x \in A$.

Examples. 1. The function $f : [0, 1] \to \mathbf{R}$ given by $f(x) = x + 2$ is a bounded function whereas $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x + 2$ is not a bounded function.

2. The function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is a bounded function.

3. The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin x$ is a bounded function since $|\sin x| \leq 1$ for all $x \in \mathbf{R}$.

Exercises

Determine which of the following functions are bounded.

(i) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3$

(ii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$

(iii) $f : [0, 2] \to \mathbf{R}$ defined by $f(x) = x^2$

(iv) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \cos x$

(v) $f : [0, \frac{\pi}{2}) \to \mathbf{R}$ defined by $f(x) = \tan x$

(vi) $f : [0, 1] \to \mathbf{R}$ defined by $f(x) = x^3 + 7x^2 + 6x - 200$

Revision Questions

I. Determine which of the following statements are true and which are false.

1. The set $A = \{x|x \in Q \text{ and } x > 0\}$ is bounded below.
2. The set $A\{x|x \in [0, 1] \text{ and } x \text{ is a rational number } \}$ is a bounded set.
3. The set Z is neither bounded above nor bounded below.
4. Let $A = \{\frac{1}{n}|n \in Z - \{0\}\}$. Then the *glb* of A is -1.
5. Let $A = \{\frac{1}{n}|n \in N\}$. Then the *glb* of A is 0.
6. Let $A = \{x|x \in \mathbf{R} \text{ and } 3 < x^2 \leq 16\}$. Then the *glb* of A is 3 and *lub* of A is 16.
7. The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 2 \sin x \cos x$ is a bounded function.
8. The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x|$ is a bounded function.
9. If A and B are bounded subsets of $\mathbf{R}$, then $A \cup B$ is a bounded subset of $\mathbf{R}$.
10. If A and B are unbounded subsets of $\mathbf{R}$, then $A \cap B$ is an unbounded subset of $\mathbf{R}$.

II. For each of the following problems, choose the correct answer.

1. Let $A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$. Let α and β be the *lub* and *glb* of A respectively. Then

   (A) $\alpha = 1$ and $\beta = 0$

   (B) $\alpha = \infty$ and $\beta = 0$

   (C) $\alpha = 1$ and $\beta = \frac{1}{2}$

   (D) $\alpha = 2$ and $\beta = \frac{1}{2}$

2. Let α and β be the *lub* and *glb* of the set $A = \{x|x \in \mathbf{R}$ and $7 < x^2 \leq 35\}$. Then

   (A) $\alpha \in A$ and $\beta \notin A$

   (B) $\alpha \notin A$ and $\beta \in A$

   (C) $\alpha \notin A$ and $\beta \notin A$

   (D) $\alpha \in A$ and $\beta \in A$

3. For which of the following sets both *lub* and *glb* are in the set

   (A) $[0, \infty)$ (B) $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$

   (C) $\{x|x \in [0,1]$ and x is irrational $\}$

   (D) $\{x|x \in [0,1]$ and x is rational$\}$

4. Which of the following subset of $\mathbf{R}$ is bounded above, but not bounded below.

   (A) N (B) Z (C) $\{\pm\frac{1}{n}|n \in N\}$ (D) $(-\infty, 1258]$

5. Let $A = \{\sin x|x \in \mathbf{R}\}$. Then the *lub* and *glb* of A are

   (A) 1 and -1 (B) 1 and 0

   (C) $\frac{1}{2}$ and $-\frac{1}{2}$ (D) 0 and -1

6. Let $A = \{x | x^2 \leq 100\}$. Then the *lub* and *glb* of A are

   (A) 100 and 0 (B) 10 and 0

   (C) 100 and -100 (D) 10 and -10

7. Let $f : [0, 100) \to \mathbf{R}$ be defined by $f(x) = \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Then the range of f is

   (A) [0, 100) (B) $\{0, 1, 2, \ldots, 100\}$

   (C) $\{1, 2, \ldots, 100\}$ (D) $\{0, 1, 2, \ldots, 99\}$

8. Which of the following is a bounded function?

   (A) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 2x$

   (B) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$

   (C) $f : (0, 1) \to \mathbf{R}$ defined by $f(x) = \frac{1}{x}$

   (D) $f : [0, 1] \to \mathbf{R}$ defined by $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

9. Which of the following function $f : (0, 1) \to \mathbf{R}$ is an unbounded function?

   (A) $f(x) = \dfrac{1}{x+1}$ (B) $f(x) = \dfrac{1}{x-1}$

   (C) $f(x) = \dfrac{x}{x+3}$ (D) $f(x) = x$

10. The range of the function $f : [-2, 2] \to \mathbf{R}$ defined by $f(x) = x^2$ is

    (A) $[-4, 4]$ (B) $[0, 2]$ (C) $[0, 4]$ (D) (0,4)