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Introduction to Probability Theory

1.0 INTRODUCTION

In our daily life, we come across many uncertainties, like weather forecasting, winning a race and lifetime of a plant in a laboratory. The outcomes of the above examples are not known in advance. Probabilities are associated with experiments where the outcomes are not known in advance or cannot be predicted. The value of probability is a number between 0 and 1, both the values being inclusive.

Probability is the branch of mathematics devoted to the study of such events. People have always been interested in games of chance and gambling. The existence of games such as dice is evident since 3000 BC. But such games were not treated mathematically till fifteenth century. During this period, the calculation and theory of probability originated in Italy. Later in the seventeenth century, French contributed to this literature of study. The foundation of modern probability theory is credited to the Russian mathematician, Kolmogorov. In 1993, he proposed the axioms on which the present subject of probability is based.

1.1 INTRODUCTION TO SET THEORY

It is very convenient to introduce probability through set theory. Most of the people who study probability have already encountered set theory and are familiar with terms such as *set, element, union, intersection,* and *complement*. In this paragraph, we introduce the basic definition and properties of sets, which are important in the study of probability.

1.1.1 Set

A well-defined collection of objects is called a *set*. These objects are called elements or members of the set. Generally, uppercase letters are used to represent a set and lowercase letters are used to represent the elements of a set.

Example 1.1.1: $A = \{a, b, c, d\}$ is a set and a, b, c and d are its elements.

Note that the elements of a set are enumerated within a pair of curly brackets as shown in this example.

Some commonly used sets and their standard notations are as follows:

1. Set of natural numbers: N
2. Set of integers: Z

3. Set of rational numbers: Q
4. Set of real numbers: R
5. Set of complex numbers: C
6. Open interval on real number line:

$$(a, b) = \{x : a < x < b, a, b \in R\}$$

7. Closed interval on real number line:

$$[a, b] = \{x : a \leq x \leq b, a, b \in R\}$$

8. Semi open intervals:

$$(a, b] = \{x : a < x \leq b, a, b \in R\}$$
$$[a, b) = \{x : a \leq x < b, a, b \in R\}$$

1.1.2 Representation of a Set

A set can be represented by two methods. They are:

i. Roster method
ii. Set builder form

i. Roster Method

Roster method is also known as tabular method. In this method, a set is represented by listing all the elements of the set, the elements being separated by commas and are enclosed within flower brackets { }. Suppose, a set containing elements 1, 2, 3, 4, 5 is represented as {1, 2, 3, 4, 5}.

ii. Set Builder Method

Set builder method is also known as property method. Instead of listing all the elements, we can represent a set in the set builder notation specifying some common property satisfied by the elements. Thus, the set $B = \{x \mid 0 \leq x \leq 7$ and x is an odd integer$\}$ or $B = \{x{:}0 \leq x \leq 7$ and x is an odd integer$\}$ represents the set {1, 3, 5, 7}. We read '$x\mid$' or 'x:' as 'all x such that'. Particularly, if a set is having infinite number of elements, listing all the elements is cumbersome; for such a case this representation is useful.

1.1.3 Finite and Infinite Sets

If a set contains finite number of elements then it is said to be a *finite set*. Otherwise, it is an *infinite set*.

$A = \{a, e, i, o, u\}$ and set of binary digits are examples of finite set. Examples of infinite set are set of rational numbers between the integers 1 and 10, and set of all points on a line segment of length 2 m.

1.1.4 Countable and Uncountable Sets

If all elements of a set can be put in one-to-one correspondence with the natural numbers, then the set is said to be *countable*. If a set is not countable, it is called *uncountable*.

1.1.5 Null Set or Empty Set or Void Set

If a set does not contain at least one element then it is said to be a null set or empty set. In other words, a set with no elements is an empty set and is denoted by ϕ or { }.

Note: $\{\phi\}$, $\{0\}$ are not null sets.

Example 1.1.2: $E = \{x : x^2 = 9 \text{ and } 3x = 6\}$ is an empty set because there is no number that can satisfy both $x^2 = 9$ and $3x = 6$.

$\therefore \quad E = \phi$.

1.1.6 Singleton Set (Singlet) and Cardinality

A set consisting of a single element is called a singleton set or singlet.

The number of elements in a finite set A is called the cardinality of A and is denoted by $n(A)$. The cardinality of the singleton set is 1. For example, consider a set $A = \{4\}$. It consist only one element, i.e. 4. Hence this set is called singleton and the cardinality of A is $n(A) = 1$.

1.1.7 Equivalent Sets

Two finite sets A and B are said to be equivalent if cardinality of both the sets are equal, i.e.

$$n(A) = n(B)$$

1.1.8 Equal Sets

Two sets A and B are said to be equal if and only if they contain the same elements, i.e. if every element of A is in B and every element of B is in A. We denote the equality by $A = B$.

Example 1.1.3: $A = \{1, 3, 5, 7, 9\}$ and $B = \{9, 1, 7, 3, 5\}$

$$A = B$$

Note: Elements in both the sets need not be in same order.

1.1.9 Universal Set

In any application of the theory of sets, the members of all sets under consideration usually belong to some fixed large set called the universal set. Usually we denote the universal set by the letter U. Suppose, if we consider dice experiment, the universal set consists of all possible outcomes, $\{1, 2, 3, 4, 5, 6\}$.

1.1.10 Subsets

A set A is called a *subset* of B (or B is called the *superset* of A), denoted by $A \subseteq B$, if all the elements of A are also elements of B. Thus $A \subseteq B$ if and only if $x \in A \Rightarrow x \in B$.

If A is a *subset* of B and there is at least one element in B which is not an element of A, then A is called a *proper subset* of B. We write $A \subset B$.

Example 1.1.4: Let $A = \{1, 2, 3, 5\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$ and $C = \{2, 4, 6, 8\}$. Then $A \subset B$, $C \not\subset B$, $B \not\subset C$, etc.

A set A is a subset of itself $A \subseteq A$.

$A \subseteq B$ and $B \subseteq C$ implies that $A \subseteq C$.

The null set ϕ is a subset of every set.

If the set A is *finite* w. .h n number of elements, then A has 2^n subsets. For example, the set of binary digits $B = \{0, 1\}$ has $2^2 = 4$ subsets.

These are ϕ, $\{0\}$, $\{1\}$ and B.

If an element x is a member of the set A, we write $x \in A$. If x is not a member of A, we write $x \notin A$. In example 1.1.1, $a \in A$ and $e \notin A$.

1.1.12 Power Set

The set of all subsets of any set S is called the power set of S. We denote the power set of S by $P(S)$.

i. Let $S = \{1, 2\}$, then $P(S) = \{\{1\}, \{2\}, \{1, 2\}, \phi\}$

ii. Let $R = \{a, b, c\}$ then $\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}, \phi\}$

If a set S is finite, say S has n elements, then the power set of S can be shown to have 2^n elements.

1.2 SOME THEOREMS ON SETS

Theorem 1: Null set is the subset of every set.

Proof: Let A be any set. In order to prove that $\phi \subseteq A$ we must show that there is no element of ϕ which is not present in A. And since ϕ contains no element at all, no such element can be found out. Hence $\phi \subseteq A$.

Theorem 2: For any set A we have $\phi \subseteq A \subseteq U$.

Proof: Every set A is a subset of universal set U since by definition all elements of A belong to U. Also the null set $\phi \subseteq A$.

Theorem 3: If A is a subset of ϕ then $A = \phi$.

Proof: The null set ϕ is a subset of every set, in particular $\phi \subseteq A$. By hypothesis, $A \subseteq \phi$. The two conditions imply $A = \phi$.

Theorem 4: The total number of all possible subsets of a given set containing n elements is 2^n.

Proof: The number of subsets each containing r elements is

$$^nC_r = \frac{n!}{r!(n-r)!}, \; r = (0, 1, 2, 3, \ldots, n)$$

Hence, the total number of subsets is

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n = 2^n$$

1.3 SET OPERATIONS

We can combine events by set operations to get other events. Following set operations are useful:

Union: The union of two sets A and B is defined as the set of elements that are either in A or in B or in both and it is denoted by $A \cup B$. In set builder notation, it can be represented as

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \qquad \ldots(1.1)$$

Intersection: The set of common elements in any two sets A and B is called intersection of the two sets. It is denoted by $A \cap B$ and can be written as

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \qquad \ldots(1.2)$$

Difference: The *difference* of two sets A and B, denoted by A/B is the set of those elements of A which do not belong to B. Thus,

$$A/B = \{x : x \in A \text{ and } x \notin B\} \qquad ...(1.3)$$

Similarly, $$B/A = \{x : x \in B \text{ and } x \notin A\} \qquad ...(1.4)$$

Complement: The complement of a set A, denoted by A^c, is defined as the set of all elements which are not in A.

Let U be a universal set and A be any subset of U, then the elements of U which are not in A, i.e. $U - A$ is the complement of A w.r.t. U and is written as $A' = U - A = A^c$.

$$A^c = \{x : x \notin A\} \qquad ...(1.5)$$

Clearly, $A^c = U/A$

Disjoint: Two sets A and B are called *disjoint* if $A \cap B = \phi$.

Example 1.3.1: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 5\}$, $B = \{2,3,4,5\}$ and $C = \{6, 8\}$. Find:

i. $A \cup B$ ii. $A \cap B$ iii. $B \cap C$ iv. A/B v. A^c

Solution: i. $A \cup B = \{1, 2, 3, 4, 5\}$ ii. $A \cap B = \{2, 3, 5\}$ iii. $B \cap C = \phi$

iv. $A/B = \{1\}$ v. $A^c = \{4, 6, 7, 8, 9\}$.

Algebraic Properties of Set Operations

Idempotent laws

If A is any set, then

i. $A \cup A = A$
ii. $A \cap A = A$

Identity laws

i. $A \cup \phi = A$
ii. $A \cap U = A$

Commutative laws

If A, B are two sets, then

i. $A \cup B = B \cup A$
ii. $A \cap B = B \cap A$

Associative laws

If A, B, C are three sets, then

i. $A \cup (B \cup C) = (A \cup B) \cup C$
ii. $A \cap (B \cap C) = (A \cap B) \cap C$

Distributive laws

If A, B, C are three sets, then

i. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
ii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's Laws

i. $(A \cup B)' = A' \cap B'$

Proof: Let $x \in (A \cup B)'$

$\Rightarrow \quad x \notin A \cup B$

$\Rightarrow \quad x \notin A$ or $x \notin B$

$\Rightarrow \quad x \in A'$ and $x \in B'$

$\Rightarrow \quad x \in A' \cap B'$

$\therefore \quad (A \cup B)' = A' \cap B'$

ii. $(A \cap B)' = A' \cup B'$

Proof: Let $x \in (A \cap B)'$

$\Rightarrow \quad x \notin A \cap B$

$\Rightarrow \quad x \notin A$ and $x \notin B$

$\Rightarrow \quad x \in A'$ or $x \in B'$

$\Rightarrow \quad x \in A' \cup B'$

$\therefore \quad (A \cap B)' = A' \cup B'$

1.4 VENN DIAGRAMS

A pictorial representation of sets by set of points in the plane is called Venn diagram. Universal set U is represented pictorially by interior of a rectangle and the other sets are represented by closed figures, viz. circles or ellipses or small rectangles or some curved figures lying within the rectangle.

The procedure to draw Venn diagrams is as follows:

Step 1: Draw the rectangle that mentions the universal set.

Step 2: Draw the circles inside the rectangle based on the conditions for indicating the sets.

Step 3: Point out the value of each circle and find out the results according to the given condition.

1.4.1 Representation of Venn Diagrams

Universal Set

Set theory is the subdivision of mathematics that learns about the sets, which are the group of objects. The universal set should be the set of all elements that under deliberation. These are represented by the capital alphabet U, sometimes E.

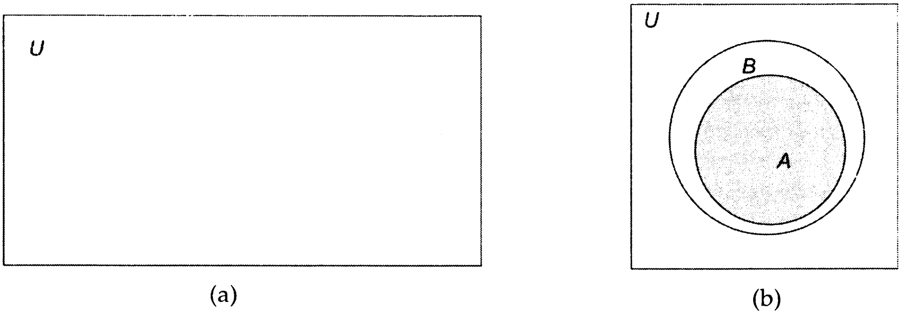


Fig.1.1: Venn diagrams: (a) Universal set (b) A is a subset of B ($A \subseteq B$)

Disjoint Sets

Let A and B be any two arbitrary sets, such that some elements are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B. We represent A and B in the pictorial form as in shown in the Venn diagram (Fig. 1.2). If the intersection of sets A and B is an empty set, then the sets A and B are called disjoint sets (Fig. 1.2a).

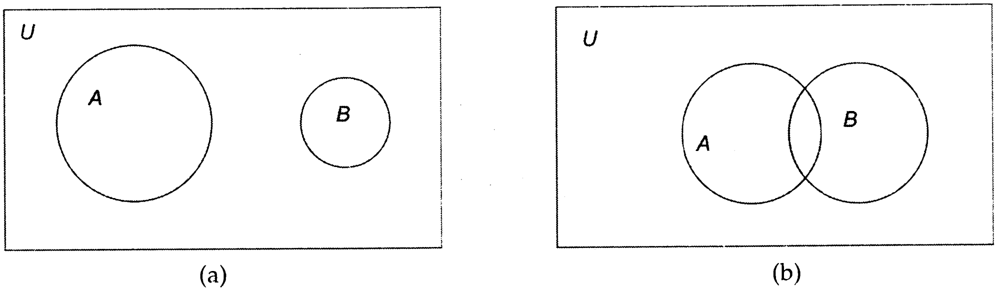


Fig. 1.2: (a) A and B are disjoint sets (b) Some elements are common in both A and B

Example 1.4 1: Given universal set, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, list the elements of the following sets:

a. $A = \{x : x \text{ is greater than } 7\}$
b. $B = \{x : x \text{ is a multiple of } 2\}$

Solution: Given $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

The elements of sets A and B should be selected from the universal set U.

a. $A = \{x : x \text{ is greater than } 7\}$
 i.e. $A = \{8, 9, 10\}$.
b. $B = \{x : x \text{ is a multiple of } 2\}$
 i.e. $B = \{2, 4, 6, 8, 10\}$.

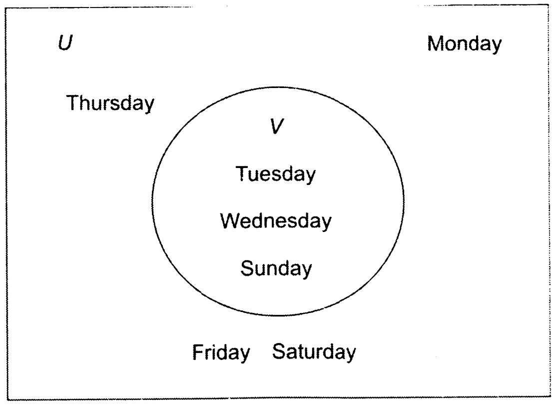
Eample 1.4.2: Draw the Venn diagram for the condition of set V = {Tuesday, Wednesday, Sunday}.

Solution: Given set V = {Tuesday, Wednesday, Sunday}

For this problem, the universal set is

U = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

The Venn diagram is as follows:



1.5 BASICS OF PROBABILITY

We introduce some of the basic concepts of probability theory by defining some terminology relating to *random experiments* (i.e. experiments whose outcomes are not predictable).

1.5.1 Outcome

The end result of an experiment is called an outcome. For example, if the experiment consists of throwing a dice, the outcome would be anyone of the six faces, 1, 2, 3, 4, 5, or 6.

1.5.2 Random Experiment

An experiment repeated under essentially homogeneous and similar conditions results in an outcome, which is unique or not unique but may be one of the several possible outcomes. When the result is unique, then the experiment is called a *deterministic experiment*. If the outcome is one of the several possible outcomes, then such an experiment is called a *random experiment* or *nondeterministic experiment.*

Consider that a voltage of V volts is applied to a simple resistive circuit. The mathematical model which describes the flow of current in the circuit is $I = \frac{V}{R}$ (Ohm's law) where R is the resistance in the circuit. This model predicts the value of I, when V and R are given.

If we repeat the above experiment a number of times, each time using the same circuit, i.e. keeping V and R as constant, we would presumably expect to observe same value of I. Thus, the above model is a deterministic model and the experiment is referred to as deterministic experiment.

Now, consider an experiment such as tossing a coin. The result of this experiment will be either tail or head. So, in some experiments, we are not able to control the value of certain variables so that the results will vary from one performance of the experiment to the next even though most of the conditions are same. These experiments are called nondeterministic or random experiments.

1.5.3 Sample Space (S)

The set of all possible outcomes of a random experiment is called the sample space associated with the random experiment. Sample space can be represented by S. A sample space may be *finite, countably infinite* or *uncountable.*

A finite or countably infinite sample space is called a *discrete sample space.* An uncountable sample space is called a *continuous sample space.*

Example 1.5.1: Find the sample space of a fair dice experiment and determine if it is continuous or discrete.

The possible 6 outcomes are:

'1' '2' '3' '4' '5' '6'

The associated sample space is $S = \{'1', '2', '3', '4', '5', '6',\}$. The outcomes are countable and finite in number. Hence, the sample space is called countably finite sample space.

Example 1.5.2: Toss a fair coin until a head is obtained. Determine if the sample space is continuous or discrete.

We may have to toss the coin any number of times before a head is obtained. Thus, the possible outcomes are H, TH, TTH, TTTH,...

The outcomes are countable but infinite in number. The countably infinite sample space is $S = \{\text{H, TH, TTH,...}\}$.

Example 1.5.3: Determine the sample space of output of a radio receiver at any time and also if it is continuous or discrete.

Suppose the output voltage of a radio receiver at any time t is a value lying between –5 V and 5 V.

The associated sample space S is given by

$$S = \{s : s \in R, -5 \leq s \leq 5\} = [-5, 5]$$

Clearly S is a uncountable sample space and hence continuous.

1.5.4 Event

An event is the outcome or a combination of outcomes of an experiment. In other words, an event is a subset of the sample space.

For example, $\{H\}$ in the experiment of tossing a coin is an event and {sum is equal to 6} in the experiment of throwing a pair of dice is an event.

Occurrence of an Event

Suppose we throw a dice. Let E be the event of a perfect square number. Then $E = \{1, 4\}$. Suppose 3 appeared on the upper most face, then we say that the event E has not occurred. E occurs only when 1 or 4 appears on the upper most face. Therefore, whenever an outcome satisfies the conditions given in the event, we say that the event has occurred. In a random experiment, if E is the event of a sample space S and w is the outcome, then we say the event E has occurred if $w \in E$.

Types of Event

Simple event: If an event has one element of the sample space, then it is called a simple or elementary event.

Example: Consider the experiment of throwing a dice, there are 6 events and all events have single element.

Compound event: If an event has more than one sample points, the event is called a compound event.

In the above example of throwing a dice, $E = \{1, 4\}$ is a compound event.

Null event (ϕ): The null event, denoted by ϕ, is an event with no sample points. Thus $\phi = \overline{S}$.

Algebra of Events

In a random experiment, considering S (the sample space) as the universal set, let A, B and C be the events of S. We can define union, intersection and complement of events and their properties on S, which is similar to those in set theory.

i. $A \cup B, B \cap A, A'$ are some events of random experiment.

ii. $A - B$ is an event, which is same as "A but not B".

iii. $A \cup B = B \cup A,\ A \cap B = B \cap A$

iv. $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$

v. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C),\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

vi. $(A \cup B)' = A' \cap B', (A \cap B)' = A' \cup B'$

Complement of an event: The complement of an event E with respect to S is the set of all the elements of S which are not in E. The complement of E is denoted by E' or E^c.

Note: In an experiment if E has not occurred then E' has occurred.

Intersection of events: The intersection of two events, A and B, is the set of all outcomes which belong to A as well as B. It is denoted by $A \cap B$ or simply (AB). The intersection of A and B is also referred to as a joint event A and B. This concept can be generalized to the case of intersection of three or more events.

Union of events: The union of two events A and B, denoted by $A \cup B$ (also written as $(A + B)$ or (A or B)) is the set of all outcomes which belong to A or B or both. This concept can be generalized to the union of more than two events.

Mutually exclusive events: Two events associated with a random experiment are said to be mutually exclusive, if both cannot occur together in the same trial.

In the experiment of throwing a dice, the events $A = \{1, 4\}$ and $B = \{2, 5, 6\}$ are mutually exclusive events. In the same experiment, the events $A = \{1, 4\}$ and $C = \{2, 4, 5, 6\}$ are not mutually exclusive because, if 4 appears on the dice, then it is favorable to both events A and C. The definition of mutually exclusive events can also be extended to more than two events. We say that more than two events are mutually exclusive, if the happening of one of these, rules out the happening of all other events. The events $A = \{1,3\}$, $B = \{5, 6\}$ and $C = \{2, 4\}$ are mutually exclusive in connection with the experiment of throwing a single dice.

Let $E_1, E_2,..., E_n$ be n events associated with a random experiment. They are said to be pairwise mutually exclusive, if $E_i \cap E_j = \phi$ for all i, j and $i \neq j$.

Mutually exhaustive events: For a random experiment, let $E_1, E_2, E_3,..., E_n$ be the subsets of the sample space S. $E_1, E_2, E_3,..., E_n$ forms a set of exhaustive events if $E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = S$.

A set of events $E_1, E_2, E_3,..., E_n$ of S are said to be mutually exclusive and exhaustive events if

$E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = S$ and

$E_i \cap E_j = \phi$ and $i \neq j$.

Equally likely outcomes: The outcomes of a random experiment are said to be equally likely, if each one of them has an equal chance of occurrence.

Example: The outcomes of an unbiased coin are equally likely.

1.6 PROBABILITY OF AN EVENT

The probability of an event has been defined in several ways. Two of the most popular definitions are the *relative frequency* definition, and the *classical* definition.

1.6.1 The Relative Frequency Definition

Suppose that a random experiment is repeated n times. If an event A occurs n_A times, then the probability of A, denoted by $P(A)$, is defined as

$$P(A) = \lim_{n\to\infty}\left(\frac{n_A}{n}\right) \qquad \text{...(1.6)}$$

where $\left(\frac{n_A}{n}\right)$ represents the fraction of occurrence of A in n trials.

1.6.2 The Classical Definition

The relative frequency definition given above has empirical flavor. In the classical approach, the probability of an event A is found without experimentation. This is done by counting the total number N of the possible outcomes of the experiment. If N_A of those outcomes are favorable to the occurrence of the event A, then

$$P(A) = \left(\frac{N_A}{N}\right) \qquad \text{...(1.7)}$$

where, it is assumed that all outcomes are *equally likely*.

Axiomatic Definition of Probability

Whatever may be the definition of probability, we require the probability measure (to the various events on the sample space) to obey the following postulates or axioms:

$$\left.\begin{array}{l}\text{i. } 0 \le P(A) \le 1 \\ \text{ii. } P(S) = 1 \\ \text{iii. If } A_1, A_2, \ldots, A_n \text{ are mutually exclusive events, then} \\ \quad P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)\end{array}\right\} \qquad \text{...(1.8)}$$

1.6.3 Theorems of Probability

Theorem 1: Addition rule of probability

If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

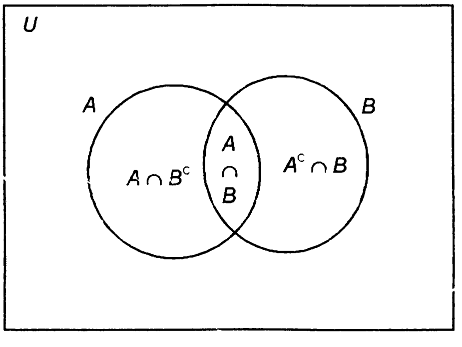


Fig. 1.3: Venn diagram of $A \cup B$

Proof: We have $A \cup B = A \cup (A^c \cap B)$ and $A \cap (A^c \cap B) = \phi$ (from the Venn diagram)

$$\therefore \qquad P(A \cup B) = P(A) + P(A^c \cap B) \qquad \text{...(1.9)}$$

($\because$ A and $A^c \cap B$ are mutually exclusive)

We have $B = (A \cap B) \cup (A^c \cap B)$ and $(A \cap B) \cap (A^c \cap B) = \phi$

$$\therefore \qquad P(B) = P(A \cap B) + P(A^c \cap B)$$

$$\therefore \qquad P(A^c \cap B) = P(B) - P(A \cap B) \qquad \text{...(1.10)}$$

Substituting Eq. (1.10) in Eq. (1.9), we get

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: If $A \cap B = \phi$ then $P(A \cup B) = P(A) + P(B)$

i.e. if A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

Theorem 2: $P(A^c) = 1 - P(A)$

Proof: $A \cap A^c = \phi$ and $A \cup A^c = S$

$$P(A \cup A^c) = P(S)$$

$$\Rightarrow \qquad P(A) + P(A^c) = 1$$

$$P(A) = 1 - P(A^c).$$

Theorem 3: $P(\phi) = 0$

Proof: The proof follows from the above Theorem 2.

$$P(\phi^c) = 1 - P(\phi)$$

$$\Rightarrow \qquad P(\phi) = 1 - P(\phi^c)$$

$$= 1 - P(S) \qquad (\because \phi^c = S)$$

$$= 1 - 1 = 0$$

Theorem 4: Given that two events A and B are statistically independent. Show that:

i. A is independent of B^c

ii. A^c is independent of B

iii. A^c is independent of B^c

Proof: i. $A = (A \cap B^c) \cup (A \cap B)$

$$P(A) = P(A \cap B^c) + P(A \cap B)$$

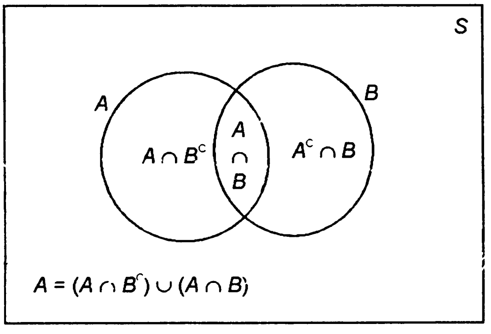


Fig. 1.4: Venn diagram of $A \cap B$

If A and B are independent,

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A) = P(A \cap B^c) + P(A) \cdot P(B)$$

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A) \cdot P(B) \\ &= P(A)\ (1 - P(B)) \\ &= P(A) \cdot P(B^c) \end{aligned}$$

Hence A is independent of B^c.

ii. $$B = (A \cap B) \cup (A^c \cap B)$$

$$P(B) = P(A \cap B) + P(A^c \cap B)$$

$$P(A^c \cap B) = P(B) - P(A \cap B)$$

$$P(A^c \cap B) = P(B) - P(A) \cdot P(B)$$

(If A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$)

$$\begin{aligned} &= P(B)(1 - P(A)) \\ &= P(B) \cdot P(A^c) \end{aligned}$$

Hence A^c and B are independent.

iii. $(A \cup B)^c = S - (A \cup B)$

$$[P(A \cup B)^c] = P(S) - P(A \cup B)$$

$$\begin{aligned} [P(A \cup B)^c] &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A) \cdot P(B) \\ &= 1 - P(A) - P(B)\ (1 - P(A)) \\ &= (1 - P(A))\ (1 - P(B)) \\ &= P(A^c)\, P(B^c). \end{aligned}$$

Hence A^c and B^c are independent.

Theorem 5:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Proof: We know that

$$P(A \cup B \cup C) = P((A \cup B) \cup C)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\begin{aligned} \therefore\ P((A \cup B) \cup C) &= P(A \cup B) + P(C) - P((A \cup B) \cap C) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P((A \cup B) \cap C) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P\{(A \cap C) \cup (B \cap C)) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - (P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)) \\ P(A \cup B \cup C) &= P(A) + P(B) + P(C) -- P(A \cap B) - (B \cap C) - P(C \cap A) + P(A \cap B \cap C). \end{aligned}$$

Theorem 6: If $B \subset A$, prove that $P(B) \leq P(A)$.

Proof: B and AB are mutually exclusive events such that $B \cup A\overline{B} = A$

$$P(B \cup AB^c) = P(A)$$

i.e. $$P(B) + P(AB^c) = P(A)$$

$$P(B) \leq P(A)$$

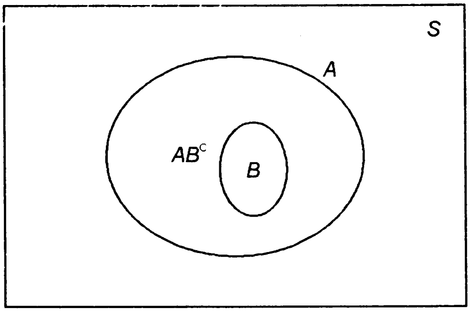


Fig. 1.5: Venn diagram of $B \cup AB^c = A$

1.7 MATHEMATICAL MODEL OF EXPERIMENTS

A real experiment is defined mathematically by three factors:

i. Assignment of a sample space
ii. Definition of events of interest
iii. Making probability assignments to events such that the axioms are satisfied.

Example 1.7.1: Three coins are tossed. Find the probability of getting:

i. Exactly two heads
ii. At least one tail
iii. Three heads

Assignment of sample space S:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Definition of events of our interest:

Let us take an event A as getting exactly two heads,
event B as getting at least one tail and
event C as getting three heads

$$n(A) = 3$$
$$n(B) = 7$$
$$n(C) = 1$$
$$P(A) = \frac{3}{8}$$
$$P(B) = \frac{7}{8}$$
$$P(C) = \frac{1}{8}.$$

1.8. JOINT AND CONDITIONAL PROBABILITY

For two events A and B, the common elements form the event $A \cap B$.

1.8.1 Joint Probability

The probability $P(A \cap B)$ is called the *joint probability* for two events A and B which intersect in the sample space. A study of Venn diagram will readily show that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad ...(1.11)$$

Equivalently,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B) \qquad ...(1.12)$$

In other words, the probability of the union of two events never exceeds the sum of the event probabilities.

1.8.2 Conditional Probability

A very useful concept in probability theory is that of *conditional probability*, denoted $P(B/A)$; it represents the probability of B occurring, given that A has occurred. In a real world random experiment, it is quite likely that the occurrence of the event B is very much influenced by the occurrence of the event A. To give a simple example, let a bowl contain 3 resistors and 1 capacitor. The occurrence of the event 'the capacitor on the second draw' is very much dependent on what has been drawn at the first instant. Such dependencies between the events are brought out using the notion of conditional probability. The conditional probability $P(B/A)$ can be written in terms of the joint probability $P(AB)$ and the probability of the event $P(A)$. This relation can be arrived at by using either the relative frequency definition of probability or the classical definition. Using the former, we have

$P(B/A)$ = Probability of occurrence of event B when the event A as already occurred.

$$P(B/A) = \frac{\text{No. of cases favourable to } A \text{ which are also favourable to } B}{\text{No. of cases favourable to } A} \qquad ...(1.13)$$

$$\therefore \qquad P(B/A) = \frac{\text{No. of cases favourable to } A \cap B}{\text{No. of cases favourable to A}} \qquad ...(1.14)$$

Also,
$$P(B/A) = \frac{\dfrac{\text{No. of cases favourable to } A \cap B}{\text{No. of cases in the sample space}}}{\dfrac{\text{No. of cases favourable to } A}{\text{No. of cases in the sample space}}}$$

$$\therefore \qquad P(B/A) = \frac{P(A \cap B)}{P(A)} \qquad ...(1.15)$$

According to relative frequency probability definition,

$$P(AB) = P(A \cap B) = \lim_{n \to \infty} \left(\frac{n_{AB}}{n} \right)$$

$$P(A) = \lim_{n \to \infty} \left(\frac{n_A}{n} \right)$$

where n_{AB} is the number of times AB occurs in n repetitions of the experiment.

As $P(B/A)$ refers to the probability of B occurring, given that A has occurred, we have conditional probability

$$P(B/A) = \lim_{n \to \infty} \left(\frac{n_{AB}}{n_A} \right)$$

$$P(B/A) = \lim_{n\to\infty}\left(\frac{\frac{n_{AB}}{n}}{\frac{n_A}{n}}\right) = \frac{P(AB)}{P(A)}, P(A) \neq 0 \qquad ...(1.16)$$

$$P(AB) = P(A \cap B) = P(B/A)\ P(A) \qquad ...(1.17)$$

Interchanging the role of A and B, we have

$$P(A/B) = \frac{P(AB)}{P(B)}, P(B) \neq 0 \qquad ...(1.18)$$

Equations (1.16) and (1.17) can be written as

$$P(A) = P(B/A)\ P(A) = P(A/B)\ P(B) \qquad ...(1.19)$$

In view of Eq. (1.19), we can also write Eq. (1.16) as

$$P(B/A) = \frac{P(B)P(A/B)}{P(A)}, P(A) \neq 0 \qquad ...(1.20)$$

Similarly

$$P(A/B) = \frac{P(A)P(B/A)}{P(B)}, P(B) \neq 0 \qquad ...(1.21)$$

Example 1.8.1: A card is drawn from an ordinary deck and we are told that it is red. What is the probability that the card is greater than 2 but less than 9.

Solution: Let A be the event of getting a card greater than 2 but less than 9.

B be the event of getting a red card. We have to find the probability of A given that B has occurred, that is, we have to find $P(A/B)$.

In a deck of cards, there are 26 red cards and 26 black cards.

$$n(B) = 26$$

Among the red cards, the number of outcomes which are favorable to A are 12, that is, $n(A \cap B) = 12$

$$\therefore \qquad P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{12}{26} = \frac{6}{13}$$

Example 1.8.2: A pair of dice is thrown. If it is known that one dice shows a 4, what is the probability that:

(a) The other dice shows a 5

(b) The total of both the dice is greater than 7.

Solution: Let A be the event that one dice shows up 4. Then the outcomes which are favourable to A are

(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (1, 4), (2, 4), (3, 4), (5, 4), (6, 4)

$$n(B) = 11$$

(a) Let B be the event of getting a 5 on one of the dice. Then the outcomes which are favourable to both A and B are (4, 4) and (4, 5)

$$n(A \cap B) = 2$$

$$\therefore \qquad P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{2}{11}$$

(b) Let C be the event of getting a total greater than 7 on both the dice.

The outcomes which are favourable to both C and A are (4, 4) (4, 5), (4, 6), (5, 4), (6, 4)

$$n(A \cap C) = 5$$

$$\therefore \qquad P(C/A) = \frac{n(A \cap C)}{n(A)} = \frac{5}{11}$$

Note that in the above example, $P(B)$ and $P(B/A)$ are different.

$\therefore$ $P(B/A)$ is $\frac{2}{11}$ whereas $P(B) = \frac{11}{36}$

Similarly, $P(C)$ and $P(C/A)$ are different.

Example 1.8.3: In a housing colony 70% of the houses are well planned and 60% of the houses are well planned and well built. Find the probability that an arbitrarily chosen house in this colony is well built given that it is well planned.

Solution: Let A be the event that the house is well planned and B be the event that the house is well built.

$$P(A) = 0.7$$

$$P(A \cap B) = 0.6$$

Probability that a house selected is well built given that it is well planned,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.6}{0.7} = \frac{6}{7}$$

1.9 STATISTICAL INDEPENDENCE

Another useful probabilistic concept is that of statistical independence. Suppose two events A and B have nonzero probabilities of occurrence, that is, assume $P(A) \neq 0$ and $P(B) \neq 0$. We call the events *statistically independent* if the probability of occurrence of one event is not affected by the other event. Mathematically,

$$P(B/A) = P(B) \qquad \text{...(1.22)}$$

Similarly

$$P(A/B) = P(A) \qquad \text{...(1.23)}$$

Equation (1.23), then

$$P(AB) = P(A \cap B) = P(A)\,P(B) \qquad \text{...(1.24)}$$

Equation (1.24) can be used to define the statistical independence of two events. Note that if A and B are independent, then $P(AB) = P(A \cap B) = P(A) \cdot P(B)$, whereas if they are disjoint, then $P(AB) = 0$. The notion of statistical independence can be generalized to the case of more than two events. A set of k events $A_1, A_2, \ldots, A_k$ are said to be statistically independent if and only if the probability of every intersection of k or fewer events equal the product of the probabilities of its constituents.

1.10 TOTAL PROBABILITY AND BAYE'S THEOREM

1.10.1 Total Probability

Theorem: (Law of total probability)

If $B_1, B_2, B_3, \ldots, B_n$ are mutually exclusive and exhaustive events of the sample space S, then for any event A of S,

$$P(A) = P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times (A/B_3) + \cdots + P(B_n) \times P(A/B_n)$$

$$= \sum_{i=1}^{n} P(B_i)P(A/B_i)$$

Proof: From the Venn diagram, we have

$$S = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n \qquad \ldots(1.25)$$

We know that

$$A = S \cap A \qquad \ldots(1.26)$$

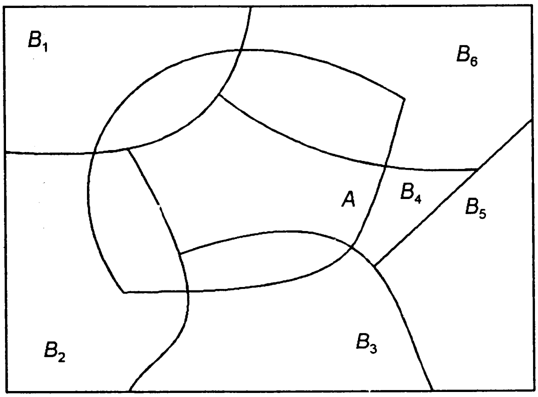


Fig. 1.6: Venn diagram showing n-mutually exclusive events B_n and another event A

Substituting Eq. (1.25) in Eq. (1.26),

$$A = (B_1 \cap A) + (B_2 \cap A) + (B_3 \cap A) + \cdots + (B_n \cap A)$$

where $B_1 \cap A$, $B_2 \cap A$, $B_3 \cap A, \ldots, B_n \cap A$ are mutually exclusive events.

$$\Rightarrow \quad P(A) = P(B_1 \cap A) + P(B_2 \cap A) + P(B_3 \cap A) + \cdots + P(B_n \cap A)$$

$$P(A) = P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3) + \cdots + P(B_n) \times P(A/B_n)$$

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i). \qquad \ldots(1.27)$$

1.10.2 Baye's Theorem

Let S be a sample space. $B_1, B_2, B_3, \ldots, B_n$ are mutually exclusive and exhaustive events such that $P(B_i) \neq 0$ for all i.

Then for any event A which is a subset of

$$S = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n \text{ and } P(A) > 0$$

we have,

$$P(B_i/A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)} \text{ for all } i = 1, 2, 3, \ldots, n \qquad \ldots(1.28)$$

Proof: We have $S = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n$ and $B_i \cap B_j = \phi$ for $i \neq j$

Since $A \subseteq S$ and $A = S \cap A$

$$= A \cap (B_1 \cup B_2 \cup B_3 \cup ... \cup B_n)$$
$$= (A \cap B_1) + (A \cap B_2) + (A \cap B_3) + \cdots + (A \cap B_n)$$
$$\therefore \quad P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + \cdots + P(A \cap B_n)$$
$$[\because (A \cap B_i) \cap (A \cap B_j) = \phi \quad \text{for } i \neq j]$$
$$P(A) = P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3) + \cdots + P(B_n) \times P(A/B_n)$$

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i) \qquad ...(1.29)$$

Also, $$P(A \cap B_i) = P(A)\, P(B_i/A) = P(B_i)\, P(A/B_i)$$

$$P(B_i/A) = \frac{P(A \cap B_i)}{P(A)} \qquad ...(1.30)$$

From Eqs. (1.29) and (1.30),

$$\Rightarrow \quad P(B_i/A) = \frac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)} \qquad ...(1.31)$$

Example 1.10.1: Let S be the sample space which is the population of adults in a small town who have completed the requirement for a college degree. The population is categorized according to sex and employment status and is as follows:

	Employed	*Unemployed*	*Total*
Male	460	40	500
Female	140	260	400
Total	600	300	900

One of these individual is to be selected for a tour throughout the country. Knowing that the individual chosen is employed, what is the probability that the individual is a man? Suppose that we are now given the additional information that 36 of those employed and 12 of those unemployed are the members of the rotary club, what is the probability of the event A that the selected individual is a member of the rotary club?

Solution: Let M be the event that a man is selected and E be the event that the individual selected is employed.

Using the reduced sample space, we have

$$P(M/E) = \text{Probability of a man who is employed} = \frac{460}{600} = \frac{23}{30}$$

Also, we have $P(M/E) = \dfrac{P(M \cap E)}{P(E)}$

From the original sample space,

$$\text{probability of a person who is employed} = P(E) = \frac{600}{900} = \frac{2}{3}$$

$$P(U) = \frac{300}{900} = \frac{1}{3}$$

$$P(M \cap E) = \frac{460}{900} = \frac{23}{45}$$

$$\Rightarrow \qquad P(M/E) = \frac{P(M \cap E)}{P(E)}$$

$$P(M/E) = \frac{23}{45} \times \frac{3}{2} = \frac{23}{30}$$

A is the event that the selected individual is a member of the rotary club.

$$P(A/E) = \frac{36}{600}$$

$$P(A/U) = \frac{12}{300}$$

According to the total probability theorem,

$$P(A) = P(E) \times P(A/E) + P(U) \times P(A/U)$$

$$P(A) = \frac{2}{3} \times \frac{3}{50} + \frac{1}{3} \times \frac{1}{25}$$

$$P(A) = \frac{4}{75}$$

Example 1.10.2: In a binary communication channel, A is the input and B is the output. $P(A) = 0.4$, $P(B/A) = 0.9$ and $P(\bar{B}/\bar{A}) = 0.6$. Find $P(A/B)$ and $P(A/\bar{B})$.

Solution: We know that

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$0.9 = \frac{P(A \cap B)}{0.4}$$

$$P(A \cap B) = 0.36$$

Given that $\quad P(\bar{B}/\bar{A}) = 0.6$

$$\frac{P(\bar{A} \cap \bar{B})}{P(\bar{A})} = 0.6$$

$$P(\bar{A} \cap \bar{B}) = (0.6) \times (0.6) \qquad [\because P(\bar{A}) = 0.6]$$

$$P(\bar{A} \cap \bar{B}) = 0.36 \qquad [\because \bar{A} \cap \bar{B} = (\overline{A \cup B})]$$

$$P(A \cup B) = 0.64$$

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.64 = 0.4 + P(B) - 0.36$$

$$P(B) = 0.6$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.36}{0.6} = 0.6$$

By the law of total probability,

$$P(A) = P(A/B)P(B) + P(A/\bar{B})P(\bar{B})$$

$$0.4 = (0.6)(0.6) + P(A/\bar{B})(0.4)$$

$$P(A/\bar{B})(0.4) = 0.4 - 0.36$$

$$P(A/\bar{B}) = \frac{0.04}{0.4} = 0.1.$$

Example 1.10.3: In a bolt factory, 25%, 35% and 40% of the total is manufactured by machines A, B and C, out of which 5%, 4% and 2% are respectively defective. If the bolt drawn is found to be defective, what is the probability that it is manufactured by the machine A, machine B, machine C?

Solution: Given $P(A) = 0.25$, $P(B) = 0.35$ and $P(C) = 0.4$

Let D be the event of getting a defective bolt.

$$P(D/A) = 0.05,\ P(D/B) = 0.04,\ P(C) = 0.02$$

$$\therefore \quad P(A/D) = \frac{P(A)P(D/A)}{P(D)} = \frac{P(A)P(D/A)}{P(A)P(D/A) + P(B)P(D/B) + P(C)P(D/C)}$$

$$= \frac{(0.25)(0.05)}{(0.25)(0.05) + (0.35)(0.04) + (0.4)(0.02)}$$

$$= \frac{0.0125}{0.0345} = \frac{125}{345}$$

$$P(B/D) = \frac{P(B)P(D/B)}{P(A)P(D/A) + P(B)P(D/B) + P(C)P(D/C)}$$

$$= \frac{(0.35)(0.04)}{(0.25)(0.05) + (0.35)(0.04) + (0.4)(0.02)}$$

$$= \frac{0.0140}{0.0345} = \frac{140}{345}$$

$$P(C/D) = \frac{P(C)P(D/C)}{P(A)P(D/A) + P(B)P(D/B) + P(C)P(D/C)}$$

$$= \frac{(0.4)(0.02)}{(0.25)(0.05)+(0.35)(0.04)+(0.4)(0.02)}$$

$$= \frac{(0.008)}{(0.25)(0.05)+(0.35)(0.04)+(0.4)(0.02)} = \frac{8}{345}.$$

EXAMPLES

Example 1.1: A dice is rolled, find the probability that an even number is obtained.

Solution: Let us first write the sample space S of the experiment.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let E be the event "an even number is obtained" and write it down.

$$E = \{2, 4, 6\}$$

We now use the formula of the classical probability,

$$P(E) = n(E)/n(S) = 3/6 = 1/2$$

Example 1.2: Two coins are tossed, find the probability that two heads are obtained. Note that each coin has two possible outcomes H (head) and T (tail).

Solution: The sample space S is given by.

$$S = \{(\text{H, T}), (\text{H, H}), (\text{T, H}), (\text{T, T})\}$$

Let E be the event "two heads are obtained"

$$E = \{(\text{H, H})\}$$

We use the formula of the classical probability,

$$P(E) = n(E)/n(S) = 1/4.$$

Example 1.3: Which of these numbers cannot be a probability?

(a) – 0.00001 (b) 0.5 (c) 1.001 (d) 0 (e) 1 (f) 20%

Solution: Probability is always greater than or equal to 0 and less than or equal to 1. Hence, only (a) and (c) above cannot represent probabilities as –0.00001 is less than 0 and 1.001 is greater than 1.

Example 1.4: Two dice are rolled; find the probability that the sum is

(a) equal to 1 (b) equal to 4 (c) less than 13.

Solution: (a) The sample space S of two dice is shown below:

$$S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6) \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

Let E be the event "sum equal to 1". There are no outcomes which correspond to a sum equal to 1, hence

$$P(E) = n(E)/n(S) = 0/36 = 0$$

(b) Three possible outcomes give a sum equal to 4: $E = \{(1, 3), (2, 2), (3, 1)\}$

Hence $\qquad P(E) = n(E)/n(S) = 3/36 = 1/12$

(c) All possible outcomes, $E = S$, give a sum less than 13, hence

$$P(E) = n(E)/n(S) = 36/36 = 1.$$

Example 1.5: A dice is rolled and a coin is tossed, find the probability that the dice shows an odd number and the coin shows a head.

Solution: The sample space S of the experiment described is as follows:

$$S = \{(1, \text{H}), (1, \text{T}), (2, \text{H}), (2, \text{T}), (3, \text{H}), (3, \text{T}), (4, \text{H}), (4, \text{T}), (5, \text{H}), (5, \text{T}), (6, \text{H}), (6, \text{T})\}$$

Let E be the event, "the dice shows an odd number and the coin shows a head". Event E may be described as follows:

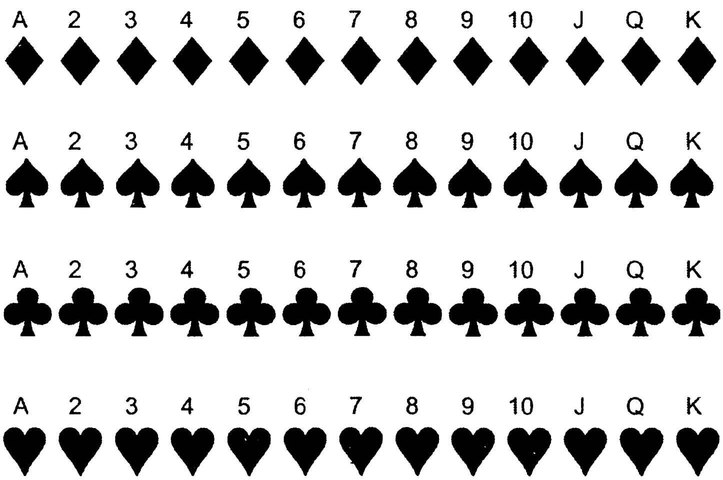
$$E = \{(1,\text{H}), (3,\text{H}), (5,\text{H})\}$$

The probability $P(E)$ is given by

$$P(E) = n(E)/n(S) = 3/12 = 1/4.$$

Example 1.6: A card is drawn at random from a deck of 52 cards. Find the probability of getting the 3 of diamond.

Solution: The sample space S of the experiment is shown below:



Let E be the event "getting the 3 of diamond". An examination of the sample space shows that there is one "3 of diamond" so that $n(E) = 1$ and $n(S) = 52$. Hence, the probability of event E occurring is given by

$$P(E) = n(E)/n(S) = 1/52.$$

Example 1.7: A card is drawn at random from a deck of 52 cards. Find the probability of getting a queen.

Solution: The sample space S of the experiment is shown above in Example 1.6.

Let E be the event "getting a queen". An examination of the sample space shows that there are 4 queens so that $n(E) = 4$ and $n(S) = 52$. Hence, the probability of event E occuring is given by

$$P(E) = n(E)/n(S) = 1/13.$$

Example 1.8: A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?

Solution: We first construct a table of frequencies that gives the marble color distributions as follows:

Color	*Frequency*
Red	3
Green	7
White	10

We now use the empirical formula of probability,

$$P(E) = \frac{\text{Frequency of white color}}{\text{Total frequency}}$$

$$P(E) = \frac{10}{20} = \frac{1}{2}.$$

Example 1.9: The blood group of 200 people is distributed as follows: 50 have A type, 65 have B type, 70 have O type and 15 have AB type. If a person from this group is selected at random, what is the probability that this person has O type blood?

Solution: We construct a table of frequencies for the blood groups as follows:

Group	*Frequency*
A	50
B	65
O	70
AB	15

We use the empirical formula of the probability,

$$P(E) = \frac{\text{Frequency of O blood}}{\text{Total frequencies}}$$

$$P(E) = \frac{70}{200} = 0.35.$$

Example 1.10: In an experiment of picking up a resistor with same likelihood of being picked up for the events; A as "draw a 47 Ω resistor", B as "draw a resistor with 5% tolerance" and C as "draw a 100 Ω resistor" from a box containing 100 resistors having resistance and tolerance as shown in table below. Determine joint and conditional probabilities.

Resistance	*Tolerance*		
	5%	*10%*	*Total*
22	10	14	24
47	28	16	44
100	24	8	32
Total	62	38	100

Solution:

$$P(A) = P(47\,\Omega) = \frac{44}{100} = \frac{11}{25}$$

$$P(B) = P(5\%) = \frac{62}{100} = \frac{31}{50}$$

$$P(C) = P(100\,\Omega) = \frac{32}{100} = \frac{8}{25}$$

Joint probabilities:

$$P(A \cap B) = P(47\,\Omega \cap 5\%) = \frac{28}{100} = \frac{7}{25}$$

$$P(B \cap C) = P(5\% \cap 100\,\Omega) = \frac{24}{100} = \frac{6}{25}$$

$$P(C \cap A) = P(100\,\Omega \cap 47\,\Omega) = 0$$

Conditional probabilities:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{7/25}{31/50} = \frac{14}{31}$$

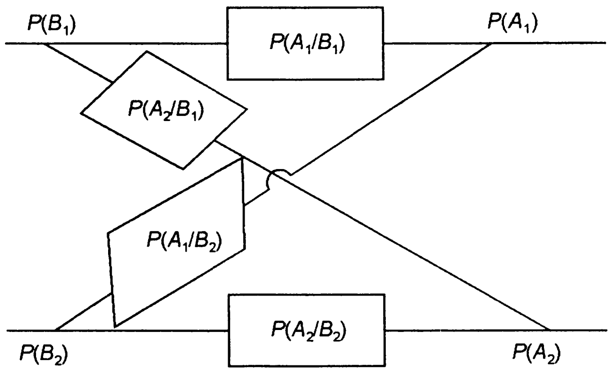
$$P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{6/25}{8/25} = \frac{3}{4}$$

$$P(C/A) = \frac{P(C \cap A)}{P(A)} = 0.$$

Example 1.11: A binary communication channel carries data as one of the two types of signals denoted by 0 and 1. Owing to noise, a transmitted 0 is sometimes received as 1 and a transmitted 1 is sometimes received as a 0. For a given channel, assume a probability of 0.95 that a transmitted 0 is correctly received as a 0 and a probability of 0.90 that a transmitted 1 is received as a 1. Further, assume a probability of 0.40 of transmitting a 0. If a signal is sent, determine the probability that:

i. A 1 is received
ii. A 1 is transmitted given that a 1 is received
iii. A 0 is received
iv. A 0 is transmitted given that a 0 is received
v. Probability of errors.

Solution:



Let event B be the input of the channel and event A be the output of the channel.

Assume B_1 if the input of the channel is zero.

Assume B_2 if the input of the channel is one.

Assume A_1 if the output of the channel is zero.

Assume A_2 if the output of the channel is one.

(a) Probability that a '1' is received is

$$P(A_2) = P(A_2/B_2)\,P(B_2) + P(A_2/B_1)\,P(B_1)$$
$$= 0.6 \times 0.9 + 0.4 \times 0.05$$
$$= 0.54 + 0.02 = 0.56$$

(b) Probability that a '1' was transmitted given that '1' was received is

$$P(B_2/A_2) = \frac{P(A_2/B_2)P(B_2)}{P(A_2)}$$

$$= \frac{0.6 \times 0.9}{0.56} = \frac{27}{28}$$

(c) Probability that a '0' is received is

$$P(A_1) = P(A_1/B_2)\,P(B_2) + P(A_1/B_1)\,P(B_1)$$
$$= 0.6 \times 0.1 + 0.4 \times 0.95$$
$$= 0.06 + 0.38 = 0.44$$

(d) Probability that a '0' was transmitted given that '0' was received is

$$P(B_1/A_1) = \frac{P(A_1/B_1)P(B_1)}{P(A_1)}$$

$$= \frac{0.4 \times 0.9}{0.44} = 0.87$$

(e) Here error means getting output is '1', even input is '0'.
Similarly getting output is '0', even input is '1'.
Getting output is '1', when input is '0',

$$P(B_1/A_2) = \frac{P(A_2/B_1)P(B_1)}{P(A_2)} = \frac{0.4 \times 0.05}{0.56} = 0.035$$

Getting output is '0', when input is '1',

$$P(B_2/A_1) = \frac{P(A_1/B_2)P(B_2)}{P(A_1)} = \frac{0.1 \times 0.6}{0.44} = 0.136.$$

Example 1.12: Roll a dice in which all faces are equally likely. What is the probability of each outcome? Find the probabilities of the events: (a) roll 4 or higher, (b) roll an even number, and (c) roll the square of an integer.

Solution: The probability of each outcome is

$$P(i) = \frac{1}{6}$$

where $\quad i = 1, 2, 3, 4, 5, 6$

The probabilities of the three events are

(a) P (roll 4 or higher) $= P(4) + P(5) + P(6) = \frac{3}{6} = \frac{1}{2}$

(b) P (roll an even number) $= P(2) + P(4) + P(6) = \frac{3}{6} = \frac{1}{2}$

(c) P (roll the square of an integer) $= P(1) + P(4) = \frac{2}{6} = \frac{1}{3}$.

Example 1.13: Four persons are to be selected from a group of 12 people, 7 of whom are women.

(a) What is the probability that the first and third selected are women?
(b) What is the probability that three of those selected are women?
(c) What is the (conditional) probability that the first and third selected are women, given that three of those selected are women?

Solution: (a) Probability that first and third selected are women is

$$P(W_1W_3) = P(W_1W_2^cW_3) + P(W_1W_2W_3)$$

$$= \left(\frac{7}{12}\cdot\frac{5}{11}\cdot\frac{6}{10}\right) + \left(\frac{7}{12}\cdot\frac{6}{11}\cdot\frac{5}{10}\right) = \frac{7}{22}$$

(b) The probability that three of those selected are women is

$$P(W_1W_2W_3) = \frac{{}^7C_3}{{}^{12}C_3} = \frac{21}{55}$$

(c) Probability that the first and third selected are women, given that three of those selected are women

$$\frac{P(W_1W_3)}{P(W_1W_2W_3)} = \frac{7/22}{21/55} = \frac{7 \times 55}{22 \times 21} = \frac{385}{462} = 0.826.$$

Example 1.14: Five boxes of random access memory chips have 100 units per box. They have respectively one, two, three, four, and five defective units. A box is selected at random, on an equally likely basis, and a unit is selected at random from there. It is defective. What are the (conditional) probabilities that the unit was selected from each of the boxes?

Solution: Let the five events be H_1, H_2, H_3, H_4, H_5.

$$P(H_1) = P(H_2) = P(H_3) = P(H_4) = P(H_5) = \frac{1}{5}$$

$$P(D/H_1) = \frac{1}{100}, P(D/H_2) = \frac{2}{100}, P(D/H_3) = \frac{3}{100},$$

$$P(D/H_4) = \frac{4}{100}, P(D/H_5) = \frac{5}{100}$$

$$P(D) = P(D/H_1)\,P(H_1) + P(D/H_2)\,P(H_2) + P(D/H_3)\,P(H_3) + P(D/H_4)\,P(H_4) + P(D/H_5)\,P(H_5)$$

$$= \frac{1}{100}\frac{1}{5} + \frac{2}{100}\frac{1}{5} + \frac{3}{100}\frac{1}{5} + \frac{4}{100}\frac{1}{5} + \frac{5}{100}\frac{1}{5} = \frac{15}{500} = \frac{3}{100}$$

$$P(H_1/D) = \frac{P(D/H_1)P(H_1)}{P(D)} = \frac{1/500}{3/100} = \frac{1}{15}$$

$$P(H_2/D) = \frac{P(D/H_2)P(H_2)}{P(D)} = \frac{2/500}{3/100} = \frac{2}{15}$$

$$P(H_3/D) = \frac{P(D/H_3)P(H_3)}{P(D)} = \frac{3/500}{3/100} = \frac{3}{15} = \frac{1}{5}$$

$$P(H_4/D) = \frac{P(D/H_4)P(H_4)}{P(D)} = \frac{4/500}{3/100} = \frac{4}{15}$$

$$P(H_5/D) = \frac{P(D/H_5)P(H_5)}{P(D)} = \frac{5/500}{3/100} = \frac{5}{15} = \frac{1}{3}.$$

Example 1.15: A bag contains 10 white and 3 black balls. Another bag contains 3 white and 5 black balls. Two balls are drawn at random from the first bag and placed in the second bag and then 1 ball is taken at random from the latter. What is the probability that it is a white ball?

Solution: The two balls transferred may be both white or both black or one white and one black.

Let event B_1 be drawing 2 white balls from the first bag,
event B_2 be drawing 2 black balls from the first bag,
event B_3 be drawing 1 white and 1 black ball from the first bag.

Clearly B_1, B_2, B_3 are mutually exclusive events.

$$P(B_1) = \frac{^{10}C_2}{^{13}C_2} = \frac{15}{26}$$

$$P(B_2) = \frac{^{3}C_2}{^{13}C_2} = \frac{1}{26}$$

$$P(B_3) = \frac{^{10}C_1 \times {^{3}C_1}}{^{13}C_2} = \frac{10}{26}$$

Assume that the two balls transferred from bag one are both white, then total balls in second bag are 10, and white balls are 5.

$$P(A/B_1) = P\text{ (drawing a white ball/2 balls have been transferred)}$$
$$= \frac{5}{10}$$

Assume that the two balls transferred from bag one are both black, then total balls in second bag are 10, and white balls are 3.

$$P(A/B_2) = P\text{ (drawing a white ball/2 balls have been transferred)}$$
$$= \frac{3}{10}$$

Assume that the two balls transferred from bag one are such that one ball is white and another is black, then total balls in second bag are 10, and white balls are 4.

$$P(A/B_3) = \text{P(drawing a white ball/2 balls have been transferred)}$$
$$= \frac{4}{10}$$

Total probability getting white ball is

$$P(A) = P(A/B_1)\, P(B_1) + P(A/B_2)\, P(B_2) + P(A/B_3)\, P(B_3)$$
$$= \frac{5}{10}\frac{15}{26} + \frac{3}{10}\frac{1}{26} + \frac{4}{10}\frac{10}{26} = \frac{59}{130}.$$

Example 1.16: Given a class of students, what is the probability that two students in the class have the same birthday?

Solution: Let $k \leq 365$ be the number of students in the class.

Then the number of possible birthdays $= 365 \cdot 365 \cdot \ldots \cdot 365$ (k times) $= 365^k$

The no. of cases with each of the k students having a different birthday is

$$= {}^{365}P_k = 365 \cdot 364 \ldots (365 - k + 1)$$

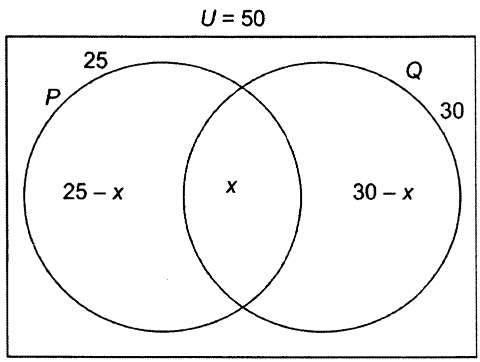
Therefore, the probability of common birthday $= 1 - \dfrac{{}^{365}P_k}{365^k}$

No. of persons	*Probability*
2	0.0027
10	0.1169
15	0.4114
25	0.5687
40	0.8912
60	0.9941
80	0.9999

Example 1.17: Out of 50 houses, 25 houses have refrigerator and 30 houses have vacuum cleane. ind how houses have both refrigerator and vacuum cleaner.

Solution: Let P = {Houses having refrigerator}

O = {Houses having vacuum cleaner}



$P \cap Q$ = {Houses having both refrigerator and vacuum cleaner}

Let $\quad n(P \cap Q) = x$

From the Venn diagram:

$$25 - x + x + 30 - x = 50$$
$$55 - x = 50$$
$$-x = 50 - 55$$
$$x = 5$$

Hence, the number of houses having both refrigerator and vacuum cleaner = 6.

Example 1.18: Verify De Morgan's law $(A \cup B)' = A' \cap B'$ using Venn diagrams.

Solution: From Fig. 1.7(ii) and (v), we find that $(A \cup B)' = A' \cap B'$.

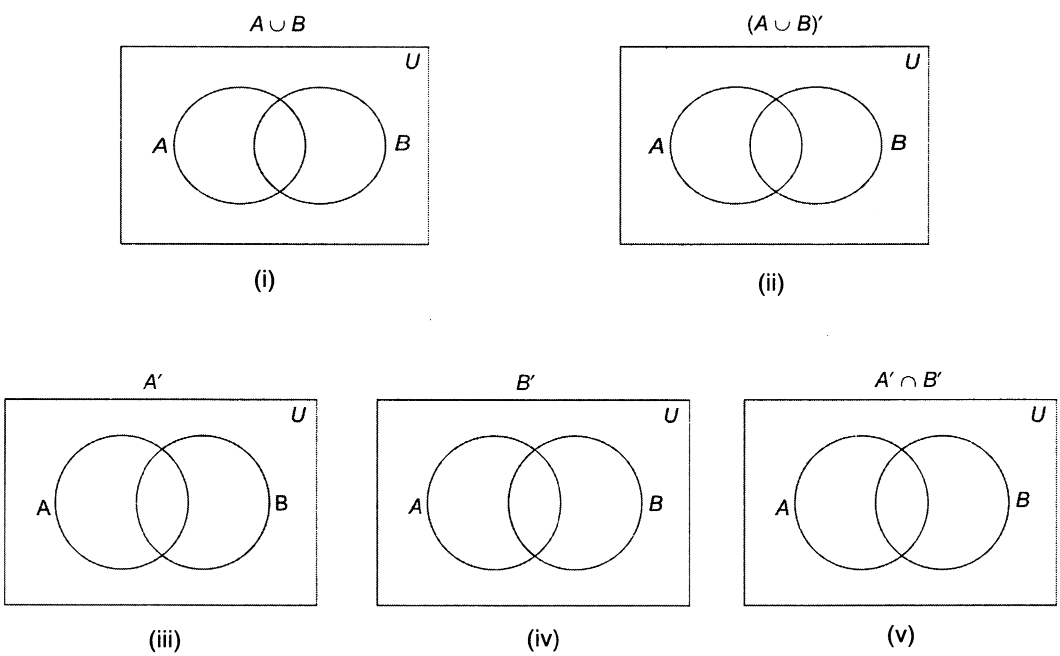


Fig. 1.7

Example 1.19: Verify the De Morgan's law $(A \cap B)' = A' \cup B'$, using Venn diagrams.

Solution: From Fig. 1.8(ii) and (v), we find that $(A \cap B)' = A' \cup B'$,

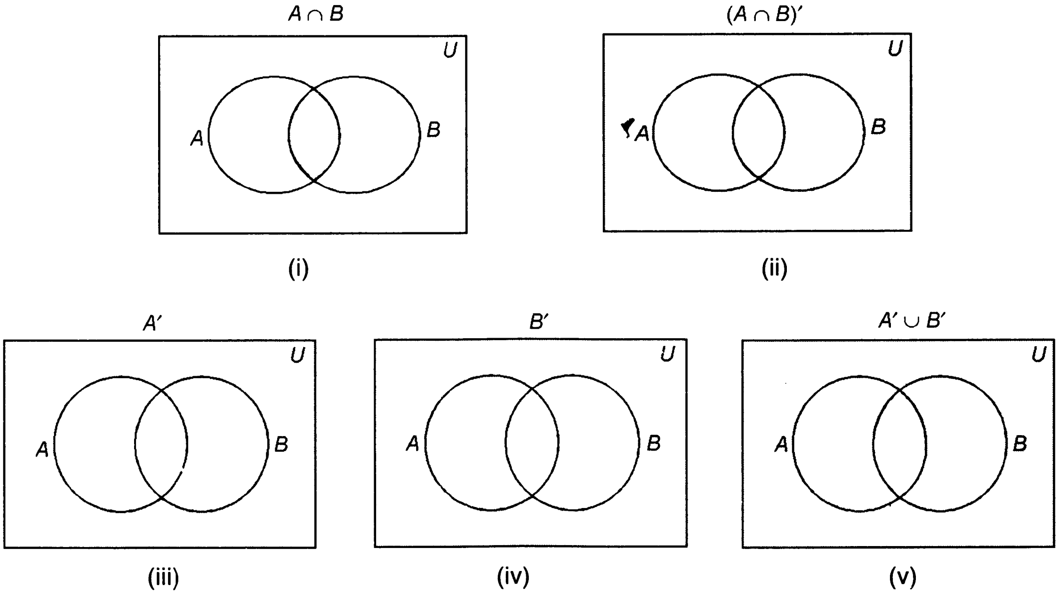


Fig. 1.8

Example 1.20: Verify the De Morgan's law $A - (B \cup C) = (A - B) \cap (A - C)$ using Venn diagrams.

Solution: From Fig. 1.9(ii) and (v), we find that $A - (B \cup C) = (A - B) \cap (A - C)$

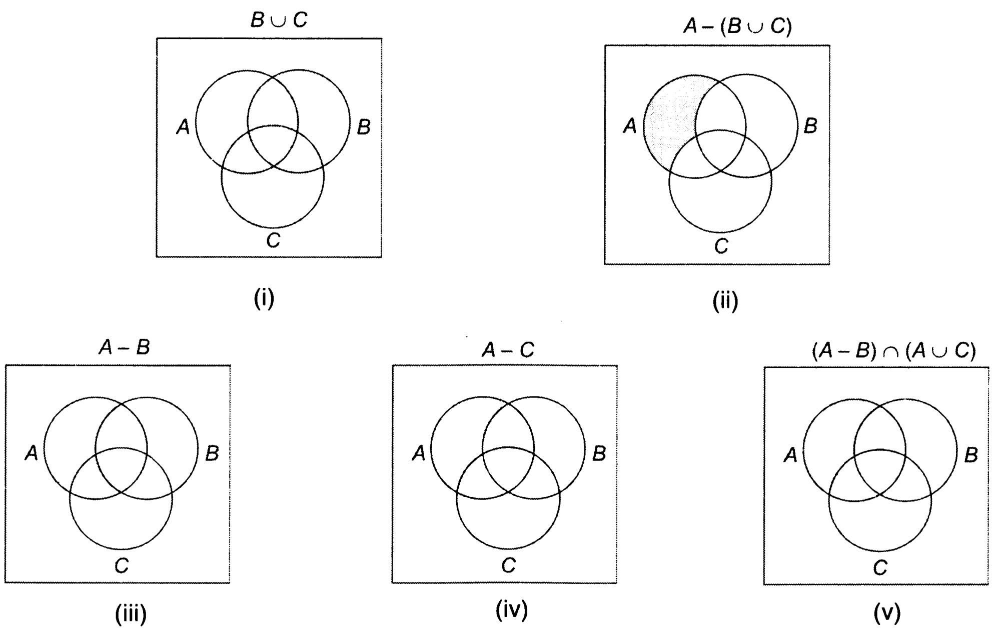


Fig. 1.9

Example 1.21: Verify the law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ using tri Venn diagrams template.

Solution: Using tri Venn diagrams, we find from Fig. 1.10(ii) and (v) that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

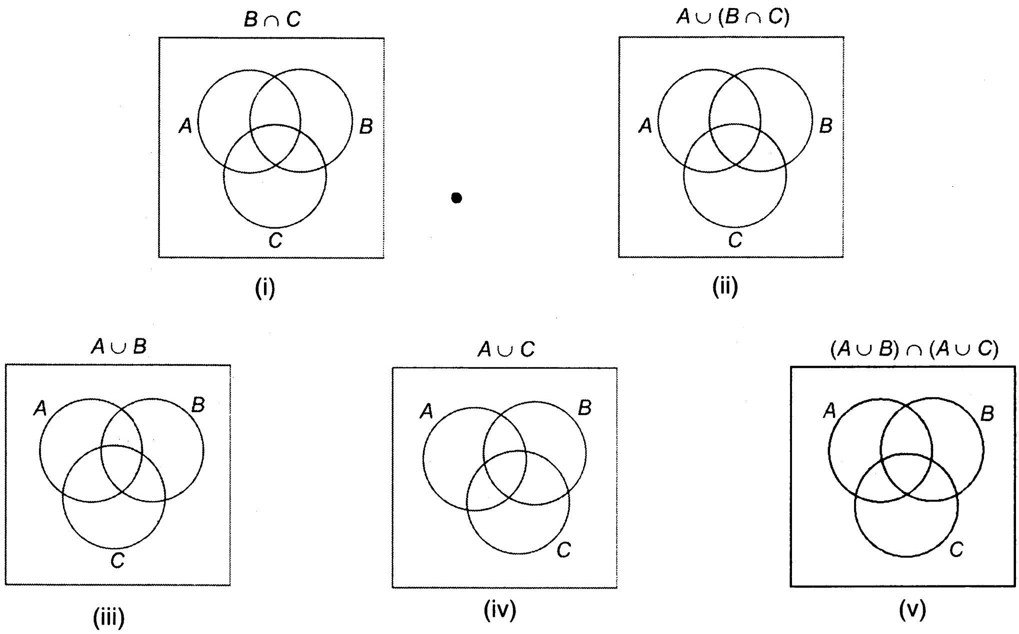
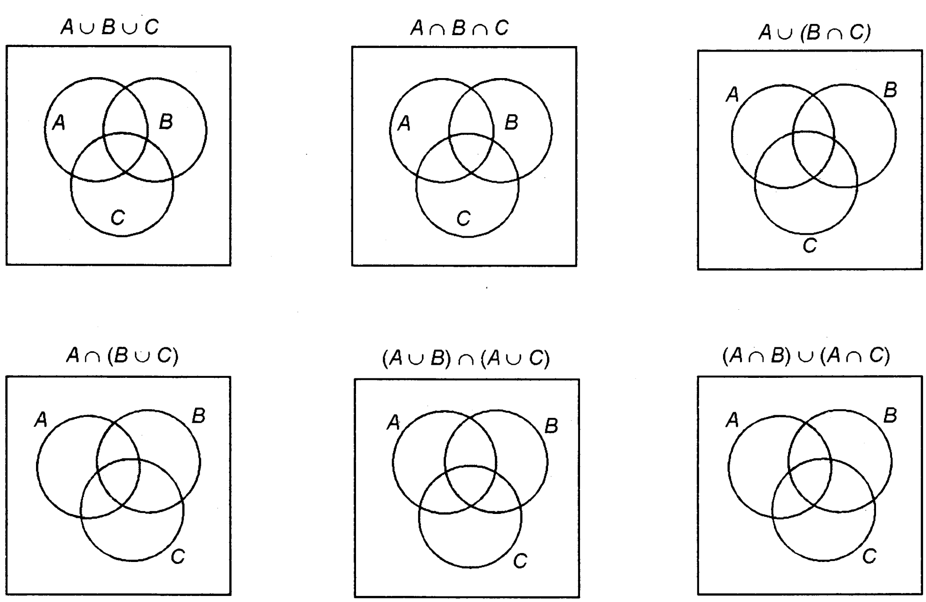


Fig. 1.10

Hence the proof.

The figures below are the representations of the three circle Venn diagrams:



EXERCISE

1. Consider the universal set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Define two subsets $A = \{x : x \leq 8\}$ and $B = \{1, 3, 5, 7, 9\}$. Find: i. A^c ii. A/B iii. $A \cap B$ iv. $A \cup B$
2. Use Venn diagram to show that:
   i. $[A/(A \cap B) \cup B] = A \cup B$
   ii. $(A \cup B \cup C)/(A \cap B \cap C) = (A^c \cap B) \cup (B^c \cap C) \cup (C^c \cap A)$
3. State whether the following statements about the subsets A, B and C of a universal set S are true (T) or false (F).
   (a) $A \cup B \cup S = A \cap B$
   (b) $A \cup B = (A^c \cap B) \cup (A \cap B^c) \cup (A \cap B)$
   (c) $A/B = B/A$
   (d) $A = (A \cap B^c) \cup (A \cap B)$
   (e) $A/B \subseteq A$
   (f) $S^c = \{\phi\}$
4. A maths teacher gave her class two tests. 25% of the class passed both the tests and 42% of the class passed the first test. What percent of those who passed the first test also passed the second test?
5. The probability that it is Friday and that a student is absent is 0.03. Since there are 5 school days in a week, the probability that it is Friday is 0.2. What is the probability that a student is absent given that today is Friday?
6. At Kennedy Middle School, the probability that a student takes technology and Spanish is 0.087. The probability that a student takes technology is 0.68. What is the probability that a student takes Spanish given that the student is taking technology?
7. At a middle school, 18% of all students play football and basketball and 32% of all students play football. What is the probability that a student plays basketball given that the student plays football?
8. In New York state, 48% of all teenagers own a skateboard and 39% of all teenagers own a skateboard and roller blades. What is the probability that a teenager owns roller blades given that the teenager owns a skateboard?
9. In New England, 84% of the houses have a garage and 65% of the houses have a garage and a back yard. What is the probability that a house has a backyard given that it has a garage?
10. A spinner has 4 equal sectors colored yellow, blue, green and red. After spinning the spinner, what is the probability of landing on each color?
11. A single dice is rolled. What is the probability of each outcome? What is the probability of getting an even number and an odd number?
12. A glass jar contains 6 red, 5 green, 8 blue and 3 yellow marbles. If a single marble is chosen at random from the jar, what is the probability of choosing (i) a red marble (ii) a green marble (iii) a blue marble (iv) a yellow marble?
13. Choose a number at random from 1 to 5. What is the probability of each outcome? What is the probability that the number chosen is even? What is the probability that the number chosen is odd?

14. (a) What is an event and explain discrete and continuous events with an example.
    (b) Discuss joint and conditional probability.
    (c) Determine the probability of a card being either red or a queen.
15. (a) Is probability relative frequency of occurrence of some event? Explain with an example.
    (b) Define an event and explain discrete and continuous event with an example.
    (c) One card is drawn from a regular deck of 52 cards. What is the probability of the card being either red or a king?
16. (a) Discuss joint and conditional probability.
    (b) When are two events said to be mutually exclusive? Explain with an example?
    (c) Determine the probability of the card being either red or a king when one card is drawn from a regular deck of 52 cards.
17. (a) Define and explain the following with an example:
        i. Equally likely events
        ii. Exhaustive events
        iii. Mutually exclusive events
    (b) Give the classical definition of probability.
    (c) Find the probability of three half-rupee coins falling all heads up when tossed simultaneously.
18. A batch of 50 items contains 10 defective items. Suppose 10 items are selected at random and tested. What is the probability that exactly 5 of the items tested are defective?
19. Two boxes are selected randomly. The first box contains 2 black balls and 3 white balls. Second box contains 4 white and 6 black balls. What is the probability of drawing a white ball?
20. Three newspapers *A*, *B* and *C* are published in a city and a survey of readers indicates the following: 20% read *A*, 16% read *B* and 14% read *C*, 8% read *A* and *B*, 5% read *A* and *C*, 2% read *A*, *B* and *C*.

    For one adult chosen at random, compute the probability that:
    i. He reads none of the papers
    ii. He reads exactly one of papers
    iii. He reads at least *A* and *B* if it is known that he reads at least one of the papers published.