Unit - I : ALGEBRA

1 Laws of Indices, Algebra and Logarithm

1.1 INTRODUCTION

Welcome dear students and aspirant engineers to the study of mathematics. Let's begin with the basic laws of indices and algebra. As you have already studied exponents, irrational numbers, surds, and much more in your school.

Nevertheless, the concepts have been discussed here to make you acquainted with basics of engineering mathematics, which would be highly useful in coming chapters too. The key to understand any topic is to go through that in detail and please make sure to practice examples.

1.2 EXPONENT

We know that x^4 is a short way of writing $x \cdot x \cdot x \cdot x$. In general, if n is a natural number or a positive integer, then

$x^n = x \cdot x \cdot x \cdot x$ ... to n factors

Here x^n is called the nth *power* of x and x is called the *base*.

The exponent n is placed at the right and above the base x.

Thus $$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

$$\left(\frac{2}{3}\right)^5 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{32}{243}$$

1.3 LAWS OF EXPONENT (FOR REAL NUMBERS)

(1) $a^m \cdot a^n = a^{m+n}$

(2) $(a^m)^n = a^{mn}$

(3) $\dfrac{a^m}{a^n} = a^{m-n}, (m > n)$

(4) $a^m b^m = (ab)^m$

Note. $a^0 = 1$ and $\dfrac{1}{a^m} = a^{-m}$

Example 1. *Simplify:*

(*i*) $2^4 \cdot 2^5$ (*ii*) $3^{\frac{1}{2}} \cdot 3^{\frac{5}{3}}$ (*iii*) $\left(3^{\frac{1}{2}}\right)^8$ (*iv*) $\dfrac{7^{\frac{1}{2}}}{7^{\frac{1}{3}}}$ (*v*) $5^{\frac{1}{4}} \cdot 7^{\frac{1}{4}}$

Solution. (*i*) $2^4 \cdot 2^5 = 2^{4+5} = 2^9$ $[\because \ a^m \cdot a^n = a^{m+n}]$

(*ii*) $3^{\frac{1}{2}} \cdot 3^{\frac{5}{4}} = 3^{\frac{1}{2}+\frac{5}{3}} = 3^{\frac{3+10}{6}} = 3^{\frac{13}{6}}$ $[\because \ a^m \cdot a^n = a^{m+n}]$

(*iii*) $\left(3^{\frac{1}{2}}\right)^8 = 3^{\frac{1}{2}\times 8} = 3^4$ $\qquad [(a^m)^n = a^{mn}]$

(*iv*) $\dfrac{7^{\frac{1}{2}}}{7^{\frac{1}{3}}} = 7^{\frac{1}{2}-\frac{1}{3}} = 7^{\frac{3-2}{6}} = 7^{\frac{1}{6}}$ $\qquad \left[\dfrac{a^m}{a^n} = a^{m-n}\right]$

(*v*) $5^{\frac{1}{4}}\cdot 7^{\frac{1}{4}} = (5\times 7)^{\frac{1}{4}} = 35^{\frac{1}{4}}$ $\qquad [\because\ a^m \,.\, b^m = (ab)^m]$ **Ans.**

Example 2. *Find* (*i*) $(49)^{\frac{1}{2}}$ (*ii*) $(125)^{\frac{1}{3}}$ (*iii*) $(32)^{\frac{3}{5}}$ (*iv*) $(16)^{\frac{3}{2}}$ (*v*) $\dfrac{(13)^{\frac{1}{2}}}{(13)^{\frac{1}{4}}}$

Solution. (*i*) $(49)^{\frac{1}{2}} = (7^2)^{\frac{1}{2}} = 7^{2\times\frac{1}{2}} = 7^1 = 7$

(*ii*) $(125)^{\frac{1}{3}} = (5^3)^{\frac{1}{3}} = 5^{3\times\frac{1}{3}} = 5^1 = 5$

(*iii*) $(32)^{\frac{3}{5}} = (2^5)^{\frac{3}{5}} = 2^{5\times\frac{3}{5}} = 2^3 = 8$

(*iv*) $(16)^{\frac{3}{2}} = (4^2)^{\frac{3}{2}} = 4^{2\times\frac{3}{2}} = 4^3 = 64$

(*v*) $\dfrac{(13)^{\frac{1}{2}}}{(13)^{\frac{1}{4}}} = (13)^{\frac{1}{2}-\frac{1}{4}} = 13^{\frac{1}{4}}$ **Ans.**

Example 3. *Simplify:*

(*i*) $(64)^{\frac{1}{2}}\cdot(8)^{\frac{1}{3}}$ (*ii*) $(27)^{\frac{2}{3}}\cdot(81)^{\frac{1}{4}}$ (*iii*) $(125)^{\frac{1}{3}}\cdot(16)^{\frac{1}{2}}$ (*iv*) $(36)^{\frac{1}{3}}\cdot(8)^{\frac{1}{2}}\cdot(25)^{\frac{1}{2}}$

Solution. (*i*) $(64)^{\frac{1}{2}}\cdot(8)^{\frac{1}{3}} = (2^6)^{\frac{1}{2}}\cdot(2^3)^{\frac{1}{3}} = 2^{6\times\frac{1}{2}}\cdot 2^{3\times\frac{1}{3}} = 2^3\cdot 2^1 = 2^{3+1} = 2^4$

(*ii*) $(27)^{\frac{2}{3}}\cdot(81)^{\frac{1}{4}} = (3^3)^{\frac{2}{3}}\cdot(3^4)^{\frac{1}{4}} = 3^{3\times\frac{2}{3}}\cdot 3^{4\times\frac{1}{4}} = 3^2\cdot 3^1 = 3^3$

(*iii*) $(125)^{\frac{1}{3}}\cdot(16)^{\frac{1}{2}} = (5^3)^{\frac{1}{3}}\cdot(2^4)^{\frac{1}{2}} = 5^{3\times\frac{1}{3}}\cdot 2^{4\times\frac{1}{2}} = 5^1\cdot 2^2 = 5\cdot 4 = 20$

(*iv*) $(16)^{\frac{1}{3}}\cdot(8)^{\frac{1}{2}}\cdot(25)^{\frac{1}{2}} = (2^4)^{\frac{1}{3}}\cdot(2^3)^{\frac{1}{2}}\cdot(5^2)^{\frac{1}{2}} = 2^{4\times\frac{1}{3}}\cdot 2^{3\times\frac{1}{2}}\cdot 5^{2\times\frac{1}{2}}$

$= 2^{\frac{4}{3}}\cdot 2^{\frac{3}{2}}\cdot 5^1 = 2^{\frac{4}{3}+\frac{3}{2}}\cdot 5 = 2^{\frac{17}{6}}\cdot 5 = 5\cdot 2^{\frac{17}{6}}$ **Ans.**

Example 4. *Simplify:* (*i*) $\dfrac{(16)^{\frac{1}{3}}}{(2)^{\frac{2}{3}}}$ (*ii*) $\dfrac{(21)^2}{(441)^{\frac{1}{3}}}$ (*iii*) $\dfrac{(35)^{\frac{2}{3}}}{(1225)}$ (*iv*) $\dfrac{(343)^{\frac{1}{3}}}{(49)^{\frac{1}{2}}}$

Solution. (*i*) $\dfrac{(16)^{\frac{1}{3}}}{(2)^{\frac{2}{3}}} = \dfrac{(2^4)^{\frac{1}{3}}}{(2)^{\frac{2}{3}}} = \dfrac{2^{4\times\frac{1}{3}}}{2^{\frac{2}{3}}} = \dfrac{2^{\frac{4}{3}}}{2^{\frac{2}{3}}} = 2^{\frac{4}{3}-\frac{2}{3}} = 2^{\frac{2}{3}}$

$(ii)\ \dfrac{(21)^2}{(441)^{\frac{1}{3}}} = \dfrac{(21)^2}{(21^2)^{\frac{1}{3}}} = \dfrac{(21)^2}{(21)^{2\times\frac{1}{3}}} = \dfrac{(21)^2}{(21)^{\frac{2}{3}}} = (21)^{2-\frac{2}{3}} = (21)^{\frac{4}{3}}$

$(iii)\ \dfrac{(35)^{\frac{2}{3}}}{(1225)} = \dfrac{(35)^{\frac{2}{3}}}{(35)^2} = (35)^{\frac{2}{3}-2} = (35)^{-\frac{4}{3}} = \dfrac{1}{(35)^{\frac{4}{3}}}$

$(iv)\ \dfrac{(343)^{\frac{1}{3}}}{(49)^{\frac{1}{2}}} = \dfrac{(7^3)^{\frac{1}{3}}}{(7^2)^{\frac{1}{2}}} = \dfrac{7^{3\times\frac{1}{3}}}{7^{2\times\frac{1}{2}}} = \dfrac{7^1}{7^1} = 7^{1-1} = 7^0 = 1$ **Ans.**

Example 5. *Solve* $4^{x+1} \,.\, 2^x = 32$.

Solution.

$$4^{x+1} \,.\, 2^x = 32$$

$$(2^2)^{x+1} \,.\, 2^x = 2^5 \quad \Rightarrow \quad 2^{2x+2} \,.\, 2^x = 2^5$$

$$2^{2x+2+x} = 2^5 \quad \Rightarrow \quad 2^{3x+2} = 2^5$$

$$3x + 2 = 5$$

$$3x = 3 \quad \text{or} \quad x = 1$$ **Ans.**

Example 6. *Solve* $2^{x+1} \,.\, 3^{y+2} = \dfrac{1}{6}$; $2^{2x+1} \,.\, 3^{3y+5} = \dfrac{1}{648}$.

Solution.

$$2^{x+1} \,.\, 3^{y+2} = \frac{1}{6} \quad \Rightarrow 2^x \,.\, 2 \,.\, 3^y \,.\, 3^2 = \frac{1}{6}$$

$$\Rightarrow \quad 2^x \,.\, 3^y \,.\, 2^1 \,.\, 3^2 = \frac{1}{2.3} \quad \Rightarrow \quad 2^x \,.\, 3^y = \frac{1}{2^1.3.2^1.3^2}$$

$$\Rightarrow \quad 2^x \,.\, 3^y = \frac{1}{2^2.3^3} \quad \Rightarrow \quad 2^x \,.\, 3^y = 2^{-2} \,.\, 3^{-3}$$

$$\Rightarrow \quad (2^x \,.\, 3^y)^3 = (2^{-2} \,.\, 3^{-3})^3$$

$$\Rightarrow \quad 2^{3x} \,.\, 3^{3y} = 2^{-6} \,.\, 3^{-9} \qquad \text{... (1)}$$

and

$$2^{2x+1} \,.\, 3^{3y+5} = \frac{1}{648}$$

$$\Rightarrow \quad 2^{2x} \,.\, 2^1 \,.\, 3^{3y} \,.\, 3^5 = \frac{1}{8 \times 81}$$

$$\Rightarrow \quad 2^{2x} \,.\, 3^{3y} \,.\, 2^1 \,.\, 3^5 = 2^{-3} \times 3^{-4}$$

$$\Rightarrow \quad 2^{2x} \,.\, 3^{3y} = 2^{-4} \times 3^{-9} \qquad \text{...(2)}$$

Dividing (1) by (2), we get

$$\frac{2^{3x}.3^{3y}}{2^{2x}.3^{3y}} = \frac{2^{-6}.3^{-9}}{2^{-4}.3^{-9}}$$

$$\Rightarrow \quad 2^x = 2^{-2} \quad \Rightarrow \quad x = -2$$

Substituting $x = -2$ in equation (2), we get

$$2^{-4} \,.\, 3^{3y} = 2^{-4} \,.\, 3^{-9}$$

$$\Rightarrow \quad 3^{3y} = 3^{-9}$$

$$\Rightarrow \quad 3y = -9$$

$$\Rightarrow \quad y = -3, \text{ and } x = -2$$ **Ans.**

1.4 IRRATIONAL NUMBERS

The numbers which can not be expressed in the form $\frac{p}{q}$ are called *irrational number*.

Ex : $\sqrt{2}, \sqrt{3}, \sqrt[3]{5}$, etc.

1.5 RADICAL AND RADICAL SIGN

The symbols $\sqrt{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }$ are *radical signs*. Thus $\sqrt{2}, \sqrt{3}, \sqrt[3]{5}$ are also called *radicals*.

The index of a root is the small number written above and to the left of the radical sign $\sqrt{\ }$. Thus, the index number of $\sqrt[3]{6}$ is 3. In square roots, the index 2 is not indicated but understood.

1.6 SURDS

Quantities of the type $\sqrt[n]{a}$, where a is a positive rational number and it is not possible to find exactly the nth root of a, are called surds of order n.

It is an irrational root of a rational number.

Ex : $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{8}$ and so on.

Note: $\sqrt{2+\sqrt{3}}$ is not a surd because $2 + \sqrt{3}$ is not a rational number.

Remark : $\sqrt[n]{a}$ fails to be a surd if

either (*i*) $a < 0$, *e.g.* $\sqrt{-5}$ is not a surd.

or (*ii*) a is not a rational number.

or (*iii*) the nth root of a is found exactly.

A surd of order 2 is called a quadratic surd and a surd of order 3 is called a cubic surd. Thus $\sqrt{3}$ is a quadratic surd and $\sqrt[3]{4}$ is a cubic surd.

Entire surd: If a surd does not contain a rational factor or term, *e.g.* $\sqrt{2}$ or $\sqrt{2}+\sqrt{3}$.

Mixed surd: If it contains a rational factor or term, *e.g.* $3\sqrt{2}$ or $2+\sqrt{3}$.

Pure surd: If it contains each term a surd, *e.g.* $\sqrt{2}+3\sqrt{5}$.

Binomial surd: If it is a binomial and contains at least one surd, *e.g.* $3+\sqrt{2}$.

Similar surds: Two or more surds are said to be similar when they can be reduced so as to have the same irrational factor. Thus $\sqrt{12}$ and $\sqrt{75}$ are similar surds, being respectively equivalent to $2\sqrt{3}$ and $5\sqrt{3}$..

Note: Surds of the same order can be multiplied, divided and compared.

Example 7. *Which is greater* $\sqrt[3]{5}$ or $\sqrt[4]{4}$?

Solution. $\sqrt[3]{5} = 5^{\frac{1}{3}}$ and $\sqrt[4]{4} = 4^{\frac{1}{4}}$

L.C.M of the denominators of indices 3 and 4 = 12.

$$\sqrt[3]{5} = 5^{\frac{1}{3}} = (5)^{\frac{4}{12}} = (5^4)^{\frac{1}{12}} = (625)^{\frac{1}{12}}$$

$$\Rightarrow \sqrt[4]{4} = 4^{\frac{1}{4}} = (4)^{\frac{3}{12}} = (4^3)^{\frac{1}{12}} = (64)^{\frac{1}{12}}$$

$$\Rightarrow (625)^{\frac{1}{12}} > (64)^{\frac{1}{12}}$$

$$\therefore \quad \sqrt[3]{5} > \sqrt[4]{4}$$ **Ans.**

EXERCISE 1.1

1. ***Find the value of the following:***

(*i*) $(128)^{\frac{3}{7}}$ **Ans.** 8 (*ii*) $(243)^{-\frac{2}{5}}$ **Ans.** $\frac{1}{9}$

(*iii*) $\dfrac{1}{(216)^{-\frac{2}{3}}}$ **Ans.** 36 (*iv*) $\left(\dfrac{8}{27}\right)^{-\frac{4}{3}}$ **Ans.** $\frac{81}{16}$

2. ***Solve the following:***

(*i*) $3^2 \,.\, 3^4$ **Ans.** 3^6 (*ii*) $4^2 \,.\, 8^3$ **Ans.** 2^{13}

(*iii*) $5^3 \,.\, 25^2$ **Ans.** 5^7 (*iv*) $49^2 \,.\, 7^3$ **Ans.** 7^7

(*v*) $(49)^{\frac{1}{2}} \cdot (16)^{\frac{1}{2}}$ **Ans.** 28 (*vi*) $(125)^{\frac{1}{3}} \cdot (25)^{\frac{1}{2}}$ **Ans.** 5^2

(*vii*) $(216)^{\frac{1}{3}} \cdot (36)^{\frac{1}{2}}$ **Ans.** 6^2 (*viii*) $(256)^{\frac{1}{4}} \cdot (32)^{\frac{1}{5}}$ **Ans.** 8

3. ***Simplify:***

(*i*) $\dfrac{(49)^{\frac{1}{2}}}{(343)^{\frac{1}{3}}}$ **Ans.** 7^0 (*ii*) $\dfrac{(27)^2}{(81)^{\frac{1}{3}}}$ **Ans.** $3^{\frac{14}{3}}$

(*iii*) $\dfrac{(125)^{\frac{1}{3}}}{(5)^{\frac{4}{3}}}$ **Ans.** $5^{-\frac{1}{3}}$ (*iv*) $\dfrac{(121)^{\frac{1}{2}}}{(11)^{\frac{3}{2}}}$ **Ans.** $11^{-\frac{1}{2}}$

(*v*) $\dfrac{(16)^{\frac{1}{3}}}{(8)^{\frac{1}{2}}} \times \dfrac{(4)^{\frac{1}{3}}}{(32)^{\frac{1}{5}}}$ **Ans.** $2^{-\frac{1}{2}}$ (*vi*) $\dfrac{(6)^{\frac{1}{3}} \times (9)^{\frac{2}{5}}}{(2) \times (3)^{\frac{2}{3}}}$ **Ans.** $\dfrac{3^{\frac{7}{15}}}{2^{\frac{2}{3}}}$

(*vii*) $\dfrac{(15)^{\frac{2}{3}}(3)^{\frac{1}{2}}}{(5)^2(9)^{\frac{1}{3}}}$ **Ans.** $\dfrac{3^{\frac{1}{2}}}{5^{\frac{4}{3}}}$ (*viii*) $\dfrac{(30)^{\frac{3}{4}} \cdot (2)^{\frac{1}{3}}}{(25)^{\frac{3}{8}} \cdot (6)^{\frac{13}{12}}}$ **Ans.** $3^{-\frac{1}{3}}$

4. ***Simplify the following:***

(*i*) $\left\{(64)^{\frac{3}{5}}\right\}^2 \times (8^3)^{-2}$ **Ans.** 4

(*ii*) $\left(\dfrac{a^x}{a^y}\right)^{x+y} \times \left(\dfrac{a^y}{a^z}\right)^{y+z} \times \left(\dfrac{a^z}{a^x}\right)^{z+x}$ **Ans.** 1

(*iii*) $\left(\dfrac{x^b}{x^c}\right)^{\frac{1}{bc}} \times \left(\dfrac{x^c}{x^a}\right)^{\frac{1}{ca}} \times \left(\dfrac{x^a}{x^b}\right)^{\frac{1}{ab}}$ **Ans.** 1

(*iv*) $\left(\dfrac{x^a}{x^b}\right)^{a^2+ab+b^2} \times \left(\dfrac{x^b}{x^c}\right)^{b^2+bc+c^2} \times \left(\dfrac{x^c}{x^a}\right)^{c^2+ca+a^2}$ **Ans.** 1

(*v*) $\left[x^{\frac{b+c}{c-a}}\right]^{\frac{1}{a-b}} \cdot \left[x^{\frac{c+a}{a-b}}\right]^{\frac{1}{b-c}} \cdot \left[x^{\frac{a+b}{b-c}}\right]^{\frac{1}{c-a}}$ **Ans.** 1

(*vi*) $\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\cdot\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\cdot\left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right)$ **Ans.** $x-y$

(*vii*) $\dfrac{2^{m+3}\cdot 3^{2m-n}\cdot 5^{m+n+3}\cdot 6^{n+1}}{6^{m+1}\cdot 10^{n+3}\cdot 15^{m}}$ **Ans.** 1

(*viii*) $\dfrac{2^{n}+2^{n-1}}{2^{n+1}-2^{n}}$ **Ans.** $\dfrac{3}{2}$

(*ix*) $\dfrac{3^{a}\cdot 3^{a^2-a}}{3^{a+1}3^{a-1}}\times\left[\dfrac{(3^3)^{\frac{\alpha}{3}}}{3^2}\right]^{-\alpha}$ **Ans.** 1

(*x*) $\left[\dfrac{(a^2b^3)^{\frac{2}{3}}\cdot(a^{-2}\cdot b^{-2}c)^{\frac{3}{2}}}{(a\cdot b)^{-\frac{2}{3}}\cdot c^{\frac{1}{2}}}\right]$ **Ans.** $\dfrac{c^3}{a\cdot b}$

(*xi*) $\left\{(x^{p})^{1-\frac{1}{p}}\right\}^{p^2+p+1}$ **Ans.** x^{p^3-1}

(*xii*) $\dfrac{1}{1+x^{a-b}+x^{a-c}}+\dfrac{1}{1+x^{b-c}+x^{b-a}}+\dfrac{1}{1+x^{c-a}+x^{c-b}}$ **Ans.** 1

(*xiii*) Show that $\dfrac{y^{-1}}{x^{-1}+y^{-1}}+\dfrac{y^{-1}}{x^{-1}-y^{-1}}=\dfrac{2xy}{y^2-x^2}$.

(*xiv*) If $a = xy^{p-1}$, $b = xy^{q-1}$ and $c = xy^{r-1}$, prove that $a^{q-r}\,b^{r-p}\,c^{p-q} = 1$.

(*xv*) If $a=\sqrt[3]{3}+\left(\sqrt[3]{3}\right)^{-1}$, prove that: $3a^3 - 9a = 10$.

(*xvi*) If $m = a^x$, $n = a^y$ and $a^z = (m^y\,.\,n^x)^z$, show that: $x\,y\,z = 1$.

(*xvii*) If $a^x = b^y = c^z$ and $b^2 = ac$, then show that: $\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{2}{y}$.

5. ***Solve the following equations:***

(*i*) $7^{x+7} = 49^{4x-7}$ **Ans.** $x = 3$

(*ii*) $2^{x+4} = 2^{x+3} + 4$ **Ans.** $x = -1$

(*iii*) $\left(\dfrac{a}{b}\right)^{4x-1}=\left(\dfrac{b}{a}\right)^{2x-5}$ **Ans.** $x = 1$

(*iv*) $9^{x-y} = 81$, $9^{x+y} = 729$ **Ans.** $x=\dfrac{5}{2}, y=\dfrac{1}{2}$

1.7 ALGEBRAIC IDENTITIES

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $a^2 - b^2 = (a + b)\,(a - b)$
4. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$
5. $(a + b)^3 = a^3 + b^3 + 3ab\,(a + b)$
   $= a^3 + b^3 + 3a^2b + 3ab^2$
6. $(a - b)^3 = a^3 - b^3 - 3ab\,(a - b)$
   $= a^3 - b^3 - 3a^2b + 3ab^2$

Type I. Factorisation of the perfect square polynomials

Formulae: (*i*) $a^2 + 2ab + b^2 = (a + b)^2$ (*ii*) $a^2 - 2ab + b^2 = (a - b)^2$

(*iii*) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(*a*) Here the first and third terms are perfect squares.

(*b*) The middle term = 2 (product of square roots of first and third terms). This method is illustrated by the following examples.

Example 8. *Factorise:*

(*i*) $4a^2 + 12ab + 9b^2$ (*ii*) $x^2 + 10x + 25$

(*iii*) $9x^2 + 24xy + 16y^2$ (*iv*) $16a^2 - 40ab + 25b^2$

(*v*) $x^2 + 5x + \dfrac{25}{4}$ (*vi*) $49x^4 - 168x^2y^2 + 144y^4$

Solution. (*i*) $4a^2 + 12ab + 9b^2 = (2a)^2 + (3b)^2 + 2(2a)(3b) = (2a + 3b)^2$ **Ans.**

(*ii*) $x^2 + 10x + 25 = (x)^2 + (5)^2 + 2(x)(5) = (x + 5)^2$ **Ans.**

(*iii*) $9x^2 + 24xy + 16y^2 = (3x)^2 + (4y)^2 + 2(3x)(4y) = (3x + 4y)^2$ **Ans.**

(*iv*) $16a^2 - 40ab + 25b^2 = (4a)^2 + (5b)^2 - 2(4a)(5b) = (4a - 5b)^2$ **Ans.**

(*v*) $x^2 + 5x + \dfrac{25}{4} = (x)^2 + \left(\dfrac{5}{2}\right)^2 + 2(x)\left(\dfrac{5}{2}\right) = \left(x + \dfrac{5}{2}\right)^2$ **Ans.**

(*vi*) $49x^4 - 168x^2y^2 + 144y^4 = (7x^2)^2 + (12y^2)^2 - 2(7x^2)(12y^2) = (7x^2 - 12y^2)^2$ **Ans.**

EXERCISE 1.2

Factorise the following:

1. $a^2 + 4ab + 4b^2$ **Ans.** $(a + 2b)^2$

2. $x^2 + 12x + 36$ **Ans.** $(x + 6)^2$

3. $16x^2 + 40xy + 25y^2$ **Ans.** $(4x + 5y)^2$

4. $4x^2 + 6x + \dfrac{9}{4}$ **Ans.** $\left(2x + \dfrac{3}{2}\right)^2$

5. $9x^4 + 24x^2y^2 + 16y^4$ **Ans.** $(3x^2 + 4y^2)^2$

6. $1 - 8ax + 16a^2x^2$ **Ans.** $(1 - 4ax)^2$

7. $a^2 + a + \dfrac{1}{4}$ **Ans.** $\left(a + \dfrac{1}{2}\right)^2$

Type II. Factorising the difference of two squares

If the given polynomial is in the form of the difference of two squares, then its two factors are

(*i*) Sum of the two square roots.

(*ii*) Difference of the two square roots.

$$a^2 - b^2 = (a + b)(a - b)$$

The method will be more clear by the following solved examples.

Example 9. *Factorise:*

(*i*) $x^2 - 4y^2$ (*ii*) $4a^2x^2 - 25b^2y^2$

(*iii*) $a^2 - x^2 - 2xy - y^2$ (*iv*) $4a^2 - 4b^2 + 4a + 1$

Solution. (*i*) $x^2 - 4y^2 = (x)^2 - (2y)^2 = (x + 2y)(x - 2y)$ **Ans.**

(*ii*) $4a^2x^2 - 25b^2y^2 = (2ax)^2 - (5by)^2$

$= (2ax + 5by)\,(2ax - 5by)$ **Ans.**

(*iii*) $a^2 - x^2 - 2xy - y^2 = a^2 - (x^2 + 2xy + y^2)$

$= a^2 - (x + y)^2 = (a + x + y)(a - x - y)$ **Ans.**

(*iv*) $4a^2 - 4b^2 + 4a + 1 = (4a^2 + 4a + 1) - 4b^2$

$= (2a + 1)^2 - (2b)^2 = (2a + 2b + 1)(2a - 2b + 1)$ **Ans.**

Example 10. *Factorise:*

(*i*) $81 - 16x^2$ (*ii*) $5 - 20x^2$ (*iii*) $81x^4 - y^4$

Solution. (*i*) $81 - 16x^2 = (9)^2 - (4x)^2 = (9 + 4x)(9 - 4x)$ **Ans.**

(*ii*) $5 - 20x^2 = 5(1 - 4x^2) = 5[(1)^2 - (2x)^2] = 5(1 + 2x)(1 - 2x)$ **Ans.**

(*iii*) $81x^4 - y^4 = (9x^2)^2 - (y^2)^2 = (9x^2 + y^2)(9x^2 - y^2)$

$= (9x^2 + y^2)[(3x)^2 - (y)^2]$

$= (9x^2 + y^2)(3x + y)(3x - y)$ **Ans.**

EXERCISE 1.3

Factorise the following:

1. $x^2 - 4y^2$ **Ans.** $(x + 2y)(x - 2y)$

2. $100 - 9z^2$ **Ans.** $(10 + 3z)(10 - 3z)$

3. $x^2 - y^2 + 2x + 1$ **Ans.** $(x + 1 + y)(x + 1 - y)$

4. $x^3 - x$ **Ans.** $x(x + 1)(x - 1)$

5. $x^2 - y^2 + 6y - 9$ **Ans.** $(x + y - 3)(x - y + 3)$

6. $9a^2 + 6a + 1 - 36z^2$ **Ans.** $(3a + 1 + 6z)(3a + 1 - 6z)$

7. $x^2 - 1 - 2a - a^2$ **Ans.** $(x + 1 + a)(x - 1 - a)$

8. $4a^2 - (2b - c)^2$ **Ans.** $(2a + 2b - c)(2a - 2b + c)$

9. $18a^2x^2 - 32$ **Ans.** $2(3ax + 4)(3ax - 4)$

(i) Expansion and Factorisation

We will use the formula (*i*) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

(*ii*) $(a + b)^3 = a^3 + b^3 + 3ab(a + b) = a^3 + b^3 + 3a^2b + 3ab^2$

(*iii*) $(a - b)^3 = a^3 - b^3 - 3ab(a - b) = a^3 - b^3 - 3a^2b + 3ab^2$

(ii) Factorisation of sum and difference of cubes

We will use the following formulae for factorisation of the sum and difference of two cubes:

(*i*) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ (*ii*) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example 11. *Factorise:* $216a^3 - 125$

Solution. $216a^3 - 125 = (6a)^3 - (5)^3$

$= (6a - 5)[(6a)^2 + (6a)5 + (5)^2]$

$= (6a - 5)(36a^2 + 30a + 25)$ **Ans.**

Example 12. *Factorise:* $a^6 - b^6$

Solution. $a^6 - b^6 = (a^3)^2 - (b^3)^2 = (a^3 + b^3)(a^3 - b^3)$

$= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2)$ **Ans.**

Example 13. *Factorise:* $a^3 - b^3 - a + b$

Solution. $a^3 - b^3 - a + b = (a^3 - b^3) - (a - b)$

$= (a - b)(a^2 + ab + b^2) - 1(a - b)$

$= (a - b)(a^2 + ab + b^2 - 1)$ **Ans.**

Example 14. Prove that: $\dfrac{0.96 \times 0.96 \times 0.96 + 0.04 \times 0.04 \times 0.04}{0.96 \times 0.96 - 0.96 \times 0.04 + 0.04 \times 0.04} = 1$

Solution. $\dfrac{0.96 \times 0.96 \times 0.96 + 0.04 \times 0.04 \times 0.04}{0.96 \times 0.96 - 0.96 \times 0.04 + 0.04 \times 0.04} = \dfrac{(0.96)^3 + (0.04)^3}{(0.96)^2 - (0.96)(0.04) + (0.04)^2}$

$= \dfrac{a^3 + b^3}{a^2 - ab + b^2}$ (where $a = 0.96$ and $b = 0.04$)

$= \dfrac{(a+b)(a^2 - ab + b^2)}{a^2 - ab + b^2}$

$= a + b = 0.96 + 0.04 = 1$ **Proved.**

EXERCISE 1.4

Factorise the following:

1. $a^3 + 27$ **Ans.** $(a + 3)(a^2 - 3a + 9)$

2. $x^3 + 64$ **Ans.** $(x + 4)(x^2 - 4x + 16)$

3. $1 - 125y^3$ **Ans.** $(1 - 5y)(1 + 5y + 25y^2)$

4. $125x^3 - 343y^3$ **Ans.** $(5x - 7y)(25x^2 + 35xy + 49y^2)$

5. $\dfrac{p^3}{343} + 8q^3$ **Ans.** $\left(\dfrac{p}{7} + 2q\right)\left(\dfrac{p^2}{49} - \dfrac{2pq}{7} + 4q^2\right)$

6. $128x^3y^3 - 250z^3$ **Ans.** $2(4xy - 5z)(16x^2y^2 + 20xyz + 25z^2)$

7. $\dfrac{991 \times 991 \times 991 + 9 \times 9 \times 9}{991 \times 991 - 991 \times 9 \times 9 \times 9}$ **Ans.** 1000

8. $\dfrac{1.03 \times 1.03 \times 1.03 - 0.03 \times 0.03 \times 0.03}{1.03 \times 1.03 + 1.03 \times 0.03 + 0.03 \times 0.03}$ **Ans.** 1

9. $\dfrac{5.01 \times 5.01 \times 5.01 - 0.01 \times 0.01 \times 0.01}{5.01 \times 5.01 + 5.01 \times 0.01 + 0.01 \times 0.01}$ **Ans.** 5

(iii) Factorisation of $a^3 + b^3 + c^3 - 3abc$

Formula: $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Proof. $a^3 + b^3 + c^3 - 3abc = (a^3 + b^3) + c^3 - 3abc = [(a + b)^3 - 3ab(a + b)] + c^3 - 3abc$

$= x^3 - 3abx + c^3 - 3abc$ [where $a + b = x$]

$= (x^3 + c^3) - 3abx - 3abc = (x + c)(x^2 - xc + c^2) - 3ab(x + c)$

$= (x + c)(x^2 - xc + c^2 - 3ab)$ ...(1)

Replacing 'x' by $(a + b)$ in (1), we get

$= (a + b + c)[(a + b)^2 - (a + b)c + c^2 - 3ab]$

$= (a + b + c)[a^2 + 2ab + b^2 - ac - bc + c^2 - 3ab]$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ **Proved.**

Example 15. *Factorise:* $a^3 + b^3 + 8c^3 - 6abc$

Solution. $a^3 + b^3 + 8c^3 - 6abc = (a)^3 + (b)^3 + (2c)^3 - 3(a)(b)(2c)$

$= (a + b + 2c)[a^2 + b^2 + (2c)^2 - ab - b(2c) - a(2c)]$

$= (a + b + 2c)(a^2 + b^2 + 4c^2 - ab - 2bc - 2ac)$ **Ans.**

Example 16. *Factorise:* $8a^3 + 27b^3 + 64c^3 - 72abc$

Solution. $8a^3 + 27b^3 + 64c^3 - 72abc = (2a)^3 + (3b^3) + (4c)^3 - 3 \times 2a \times 3b \times 4c$

$= (2a + 3b + 4c)\{(2a)^2 + (3b)^2 + (4c)^2 - 2a \times 3b - 3b \times 4c - 4c \times 2a\}$

$= (2a + 3b + 4c)(4a^2 + 9b^2 + 16c^2 - 6ab - 12bc - 8ca)$ **Ans.**

Example 17. *Prove that:*

$$(a + b)^3 + (b + c)^3 + (c + a)^3 - 3(a + b)(b + c)(c + a) = 2(a^3 + b^3 + c^3 - 3abc)$$

Solution. We know that $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$\therefore\ (a + b)^3 + (b + c)^3 + (c - a)^3 - 3(a + b)(b + c)(c + a)$

$= [(a + b) + (b + c) + (c + a)][(a + b)^2 + (b + c)^2 + (c + a)^2 - (a + b)(b + c)$
$- (b + c)(c + a) - (c + a)(a + b)]$

$= (2a + 2b + 2c)[(a^2 + 2ab + b^2) + (b^2 + 2bc + c^2) + (c^2 + 2ca + a^2) - (ab + ac + b^2 + bc)$
$- (bc + ba + c^2 + ca) - (ca + cb + a^2 + ab)]$

$= 2(a + b + c)(2a^2 + 2b^2 + 2ab + 2bc + 2ca - ab - ac - b^2 - bc - bc - ba - c^2 - ca - ca - a^2 - ab)$

$= 2(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= 2(a^3 + b^3 + c^3 - 3abc)$ **Proved.**

EXERCISE 1.5

Factorise the following:

1. $8a^3 + 27b^3 + c^3 - 18abc$ **Ans.** $(2a + 3b + c)(4a^2 + 9b^2 + c^2 - 6ab - 3bc - 2ac)$
2. $a^3 + 64b^3 - c^3 + 12abc$ **Ans.** $(a + 4b - c)(a^2 + 16b^2 + c^2 - 4ab + 4bc + ac)$
3. $1 - a^3 - b^3 - 3ab$ **Ans.** $(1 - a - b)(1 + a^2 + b^2 + a + b - ab)$
4. $p^3 - 125q^3 + 1 + 15pq$ **Ans.** $(p - 5q + 1)(p^2 + 25q^2 + 1 + 5pq - p + 5q)$
5. $(l + m)^3 + (2m + 3n)^3 + (3n + 4l)^3 - 3(l + m)(2m + 3n)(3n + 4l)$
   Ans. $(5l + 3m + 6n)(13l^2 + 3m^2 + 9n^2 - 12lm + 6ln)$
6. $(x + y)^3 + (y - z)^3 + (z - x)^3 - 3(x + y)(y - z)(z - x)$
   Ans. $2y(3x^2 + y^2 + 3z^2 + 3xy - 3yz - 3zx)$
7. $(p + q)^3 - (q + r)^3 + (r + p)^3 + 3(p + q)(q + r)(r + p)$
   Ans. $2p(p^2 + 3q^2 + 3r^2 + 3pr + 3qr + 3pq)$

(*iv*) Factorisation of $a^3 + b^3 + c^3$ when $a + b + c = 0$

Formula: $a^3 + b^3 + c^3 = 3abc$, when $a + b + c = 0$

Proof. We know that $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

If $a + b + c = 0$, then

$a^3 + b^3 + c^3 - 3abc = 0 \times (a^2 + b^2 + c^2 - ab - bc - ca)$

or $a^3 + b^3 + c^3 - 3abc = 0$

$a^3 + b^3 + c^3 = 3abc$ **Proved.**

We will use this formula for factorisation.

Example 18. *Factorise:* $(p - q)^3 + (q - r)^3 + (r - p)^3$

Solution. $(p - q)^3 + (q - r)^3 + (r - p)^3$ ...(1)

Put $p - q = a,\ q - r = b,\ r - p = c$ in (1), we get

$(p - q)^3 + (q - r)^3 + (r - p)^3 = a^3 + b^3 + c^3$

Now $a + b + c = p - q + q - r + r - p = 0$

$\therefore$ $a^3 + b^3 + c^3 = 3abc$ ... (2)

Putting the value of a, b, c in (2), we get

$(p - q)^3 + (q - r)^3 + (r - p)^3 = 3(p - q)(q - r)(r - p)$ **Ans.**

Example 19. *Factorise:* $(x^2 - y^2)^3 + (y^2 - z^2)^3 + (z^2 - x^2)^3$

Solution. $(x^2 - y^2)^3 + (y^2 - z^2)^3 + (z^2 - x^2)^3$ ... (1)

Put $x^2 - y^2 = a$, $y^2 - z^2 = b$, $z^2 - x^2 = c$ in (1), we get

$(x^2 - y^2)^3 + (y^2 - z^2)^3 + (z^2 - x^2)^3 = a^3 + b^3 + c^3$

Now $a + b + c = x^2 - y^2 + y^2 - z^2 + z^2 - x^2 = 0$

$\therefore$ $a^3 + b^3 + c^3 = 3abc$ ... (2)

Putting the values of a, b, c in (2), we get

$(x^2 - y^2)^3 + (y^2 - z^2)^3 + (z^2 - x^2)^3 = 3(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$ **Ans.**

EXERCISE 1.6

Factorise the following:

1. $(a - 2b)^3 + (2b - 3c)^3 + (3c - a)^3$ **Ans.** $3(a - 2b)(2b - 3c)(3c - a)$
2. $(2x - 4y)^3 + (4y - 3z)^3 + (3z - 2x)^3$ **Ans.** $3(2x - 4y)(4y - 3z)(3z - 2x)$
3. $(3p - q)^3 + (q + 2r)^3 - (2r + 3p)^3$ **Ans.** $- 3(3p - q)(q + 2r)(3p + 2r)$
4. $(l + 3m)^3 - (3m - 4n)^3 - (4n + l)^3$ **Ans.** $3(l + 3m)(3m - 4n)(4n + l)$

1.8 LOGARITHM

There is a great relation between logarithms and exponents.

If $a^x = n$ ...(1)

(where a is a positive real number, $a \neq 1$ and n is also real number)

Then, we rewrite the above statement (1) as $x = \log_a n$ ... (2)

In words, we say 'x is the logarithm of n to the base a' or x equals log n to the base a.

Relation (1) and (2) are equivalent relations.

Relation (1) is in the index form. Relation (2) is in the log form.

Note. (1) log is the short form of logarithm.

(2) Negative numbers and zero have no logarithms, log (–2) and log 0 are meaningless.

Illustration. Consider the following table

$2^4 = 16$	log of 16 to the base 2 = 4	$\log_2 16 = 4$
$3^3 = 27$	log of 27 to the base 3 = 3	$\log_3 27 = 3$
$4^{-3} = -\frac{1}{64}$	log of $y^2 \frac{1}{64}$ to the base 4 = –3	$\log_4 \frac{1}{64} = -3$
$10^{-1} = \frac{1}{10} = 0.1$	log of 0.1 to the base 10 = –1	$\log_{10} 0.1 = -1$
$a^0 = 1$	log of 1 to the base a = 0	$\log_a 1 = 0$
$a^1 = a$	log of a to the base a = 1	$\log_a a = 1$

We find two important result in the above illustrations:

(*i*) The logarithm of 1 to any base in '0', *i.e.* $\log_a 1 = 0$

(*ii*) The logarithm of any quantity to the same base in, '1' *i.e.* $\log_a a = 1$

Example 20. *Write the following in the form of logarithms:*

(*i*) $2^7 = 128$ (*ii*) $10^2 = 100$ (*iii*) $3^5 = 243$

Solution. (*i*) The logarithmic form of 2^7 ($2^7 = 128$) is $\log_2 128 = 7$

(*ii*) The logarithmic form of 10^2 ($10^2 = 100$) is $\log_{10} 100 = 2$

(*iii*) The logarithmic form of 3^5 ($3^5 = 243$) is $\log_3 243 = 5$

Example 21. *Write the following in the form of logarithms.*

(*i*) $7^4 = 2401$ (*ii*) $10^{-3} = 0.001$ (*iii*) $11^2 = 121$

Solution. (*i*) The logarithmic form of ($7^4 = 2401$) is

$$\log_7 2401 = 4$$

(*ii*) The logarithmic form of ($10^{-3} = 0.001$) is

$$\log_{10} 0.001 = -3$$

(*iii*) The logarithmic form of ($11^2 = 121$) is **Ans.**

$$\log_{11} 121 = 2.$$

Example 22. *Express each of the following in exponential form*

(i) $\log_5 125 = 3$ (*ii*) $\log_3 81 = 4$ (*iii*) $\log_{10} 10000 = 4$

(*iv*) $\log_2 256 = 8$ (*v*) $\log_6 36 = 2$ (*vi*) $\log_{10} 0.1 = -1$

Solution. We know that $\log_a b = x$ can be written in exponential from as $a^x = b$. Then

(*i*) $\log_5 125 = 3 \Rightarrow 5^3 = 125$

(*ii*) $\log_3 81 = 4 \Rightarrow 3^4 = 81$

(*iii*) $\log_{10} 10000 = 4 \Rightarrow 10^4 = 10000$

(*iv*) $\log_2 256 = 8 \Rightarrow 2^8 = 256$

(*v*) $\log_6 36 = 2 \Rightarrow 6^2 = 36$

(*vi*) $\log_{10} 0.1 = -1 \Rightarrow 10^{-1} = 0.1$ **Ans.**

Example 23. *Find the value of each of the following:*

(*i*) $\log_3 27$ (*ii*) $\log_7 \sqrt[3]{7}$

Solution. (*i*) We know from the definition that

$$\log_a b = x \Leftrightarrow a^x = b,\ a > 0,\quad a \neq 1$$

Now, let $x = \log_3 27 \Leftrightarrow 3^x = 27 \Rightarrow 3^x = (3)^3 \Rightarrow x = 3$

Therefore, $x = 3$. Hence, $\log_3 27 = 3$ **Ans.**

(*ii*) We know from the definition that

$$\log_a b = x \Leftrightarrow a^x = b,\ a > 0,\quad a \neq 1$$

Now, let $\log_7 \sqrt[3]{7} = x \Leftrightarrow 7^x = \sqrt[3]{7} \Rightarrow 7^x = (7)^{\frac{1}{3}} \Rightarrow x = \dfrac{1}{3}$

Therefore, $\log_7 \sqrt[3]{7} = \dfrac{1}{3}$. **Ans.**

Example 24. *If* $\log_{81} x = \dfrac{3}{2}$, *then find the value of* x.

Solution. We have, $\log_{81} x = \dfrac{3}{2} \Rightarrow (81)^{\frac{3}{2}} = x \Rightarrow (9^2)^{\frac{3}{2}} = x \Rightarrow 9^3 = x$

$\Rightarrow$ $x = 9 \times 9 \times 9 = 729$ **Ans.**

EXERCISE 1.7

Write the following in the form of logarithms:

1. $2^6 = 64$ **Ans.** $\log_2 64 = 6$

2. $4^5 = 1024$ **Ans.** $\log_4 1024 = 5$

3. $3^4 = 81$ **Ans.** $\log_3 81 = 4$

4. $4^3 = 64$ **Ans.** $\log_4 64 = 3$

5. $7^2 = 49$ **Ans.** $\log_7 49 = 2$

6. $8^3 = 512$ **Ans.** $\log_8 512 = 3$

7. $9^{\frac{5}{2}} = 243$ **Ans.** $\log_9 243 = \dfrac{5}{2}$

8. $10^0 = 1$ **Ans.** $\log_{10} 1 = 0$

Express each of the following in exponential form:

9. $\log_5 1 = 0$ **Ans.** $5^0 = 1$

10. $\log_{10} 1000 = 3$ **Ans.** $10^3 = 1000$

11. $\log_4 64 = 3$ **Ans.** $4^3 = 64$

12. $\log_7 343 = 3$ **Ans.** $7^3 = 343$

13. $\log_{10} 0.001 = -3$ **Ans.** $10^{-3} = 0.001$

14. $\log_3 \dfrac{1}{9} = -2$ **Ans.** $3^{-2} = \dfrac{1}{9}$

15. $\log_8 4 = \dfrac{2}{3}$ **Ans.** $8^{\frac{2}{3}} = 4$

16. $\log_9 6561 = 4$ **Ans.** $9^4 = 6561$

Find the value of each of the following by the definition of logarithm:

17. $\log_2 16$ **Ans.** 4

18. $\log_2 \sqrt{32}$ **Ans.** $\dfrac{5}{2}$

19. $\log_{10} 10^5$ **Ans.** 5

20. $\log_n 1$ **Ans.** 0

1.9 LAWS OF LOGARITHM

In this section, we shall learn the following laws of logarithm. These laws hold for any base $a (a > 0$ and $a \neq 1)$.

(i) First Law (Product Law) $\boxed{\log_a(mn) = \log_a m + \log_a n}$.

The logarithm of the product of two numbers is equal to the sum of their logarithms with reference to the same base.

Proof: Let $\log_a m = x$ and $\log_a n = y$. Then,

$$\log_a m = x \quad \Rightarrow \quad a^x = m \qquad \text{... (1)}$$

and $$\log_a n = y \quad \Rightarrow \quad a^y = n \qquad \text{... (2) [By definition]}$$

On multiplication of (1) and (2), we get

$$\therefore \quad m \cdot n = a^x \cdot a^y \quad \Rightarrow \quad m \cdot n = a^{x+y} \qquad \text{[By laws of indices]}$$

$$\Rightarrow \quad \log_a mn = x + y \qquad \text{... (3) [By definition of log]}$$

On putting values of x and y in (3), we get

$$\log_a mn = \log_a m + \log_a n$$

or $$\log 2 \times 3 = \log 2 + \log 3$$

(ii) Second Law (Quotient Law) $\boxed{\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n}$

The logarithm of quotient of two numbers is equal to the difference of logarithm of the numerator and the logarithm of the denominator.

Proof. Let $\log_a m = x$ and $\log_a n = y$. Then,

$$\log_a m = x \quad \Rightarrow \quad a^x = m \qquad \text{... (1)}$$

and $$\log_a n = x \quad \Rightarrow \quad a^y = n \qquad \text{... (2) [By definition of log]}$$

On dividing (1) by (2), we get

$$\therefore \qquad \frac{a^x}{a^y} = \frac{m}{n} \qquad \Rightarrow \qquad \frac{m}{n} = a^{x-y} \qquad \text{[By laws of indices]}$$

$$\log_e\left(\frac{m}{n}\right) = x - y \qquad \text{... (3)}$$

On putting value of $x = \log_a m$ and $y = \log_a n$ in (3), we get

$$\log a\left(\frac{m}{n}\right) = \log_a m - \log_a n \qquad \textbf{Proved.}$$

$$\text{or} \qquad \log\left(\frac{3}{4}\right) = \log 3 - \log 4$$

(iii) Third Law (Power Law) $\boxed{\log_a m^n = n\log_a m}$

The logarithm of a number raised to a power n is n times the logarithm of the number.

Proof. Let $\log_a m = x$. Then, $a^x = m$

Now, $\qquad a^x = m \qquad \Rightarrow \qquad (a^x)^n = m^n \qquad \Rightarrow \qquad a^{xn} = m^n$

$\Rightarrow \qquad m^n = a^{nx} \qquad \Rightarrow \qquad \log_a m^n = nx \qquad$ [By definition of log]

$\Rightarrow \qquad \log_a m^n = n \log_a m \qquad (\because x = \log_a m)$

Hence, $\qquad \log_a m^n = n\log_a m \qquad$ **Proved.**

For example, $\log m^2 = 2\log m$

Example 25. *Find the value:* $\log_3 27\sqrt{729}$

Solution. We have,

$$\log_3 27\sqrt{729} = \log_3 27 + \log_3\sqrt{729} = \log_3(3)^3 + \log_3(729)^{\frac{1}{2}} \qquad [\because \sqrt{a} = a^{\frac{1}{2}}]$$

$$= 3\log_3 3 + \frac{1}{2}\log_3(729) \qquad [\because \log_a m^n = n\log_a m]$$

$$= 3(1) + \frac{1}{2}\log_3(3^6) = 3 + \frac{1}{2}\cdot 6\log_3 3$$

$$= 3 + \frac{1}{2}\cdot 6(1) = 3 + 3 = 6 \qquad [\because \log_3 3 = 1]$$

Example 26. *Show that:* $3\log 4 - 2\log 6 + \log(18)^{\frac{3}{2}} = \log(96\sqrt{2})$

Solution.

$$\text{L.H.S.} = 3\log 4 - 2\log 6 + \log(18)^{\frac{3}{2}}$$

$$= \log 4^3 - \log 6^2 + \log(18)^{\frac{3}{2}} = \log 64 - \log 36 + \log(2 \times 3^2)^{\frac{3}{2}}$$

$$= \log\frac{64 \times (2 \times 3^2)^{\frac{3}{2}}}{36} = \log\frac{64 \times 2^{\frac{3}{2}} \times 3^{2 \times \frac{3}{2}}}{36} = \log\frac{64 \times 2\sqrt{2} \times 27}{36}$$

$$= \log(16 \times 2\sqrt{2} \times 3) = \log(96\sqrt{2}) = \text{R.H.S.} \qquad \textbf{Proved.}$$

Example 27. *Prove that:*

(*i*) $\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3$ (*ii*) $\log\frac{a^2}{bc} + \log\frac{b^2}{ac} + \log\frac{c^2}{ab} = 0$

(*iii*) $\log_b a \times \log_c b \times \log_a c = 1$

Solution. (*i*) L.H.S. = log (1 + 2 + 3) = log 6 = log (1 × 2 × 3)

= log 1 + log 2 + log 3 = R.H.S.

(*ii*) $$\text{L.H.S.} = \log\frac{a^2}{bc} + \log\frac{b^2}{ac} + \log\frac{c^2}{ab} = \log\left(\frac{a^2}{bc} \times \frac{b^2}{ac} \times \frac{c^2}{ab}\right)$$

$$= \log\left(\frac{a^2 \times b^2 \times c^2}{a^2 \times b^2 \times c^2}\right) = \log 1 = 0 = \text{R.H.S.} \qquad [\because \log 1 = 0]$$

(*iii*) $$\text{L.H.S.} = \log_b a \times \log_c b \times \log_a c$$

$$= \frac{\log_a a}{\log_a b} \times \frac{\log_a b}{\log_a c} \times \frac{\log_a c}{\log_a a} = 1 = \text{R.H.S.}$$ **Proved.**

(Expressing each logarithm to the same base, say a, by using base changing formula).

Example 28. *Show that:* $\log\frac{2}{3} + \log\frac{25}{49} + \log\frac{21}{2} + \log\frac{7}{5} = \log 5$ *by two methods.*

Solution. **First method**

$$\log\frac{2}{3} + \log\frac{25}{49} + \log\frac{21}{2} + \log\frac{7}{5} = \log\frac{2}{3} \times \frac{25}{49} \times \frac{21}{2} \times \frac{7}{5}$$

$$= \log\frac{2 \times 5 \times 5 \times 7 \times 3 \times 7}{3 \times 7 \times 7 \times 2 \times 5} = \log 5$$ **Proved.**

Second method

$$\log\frac{2}{3} + \log\frac{25}{49} + \log\frac{21}{2} + \log\frac{7}{5}$$

$= \log 2 - \log 3 + \log 25 - \log 49 + \log 21 - \log 2 + \log 7 - \log 5$

$= \log 2 - \log 3 + \log 5^2 - \log 7^2 + \log 3 \times 7 - \log 2 + \log 7 - \log 5$

$= \log 2 - \log 3 + 2 \log 5 - 2 \log 7 + \log 3 + \log 7 - \log 2 + \log 7 - \log 5$

$= \log 5$ **Proved.**

Example 29. *Solve* $\log(x + 4) - \log(3x - 2) = 0$

Solution. $\log(x + 4) - \log(3x - 2) = 0$

$$\log\left(\frac{x+4}{3x-2}\right) = 0$$

$$\log\left(\frac{x+4}{3x-2}\right) = \log 1$$

$$\frac{x+4}{3x-2} = 1 \quad \text{or} \quad x + 4 = 3x - 2$$

$2x = 6$ or $x = 3$ **Ans.**

Example 30. *If* $\frac{\log x}{\log 3} = \frac{\log 81}{\log 9}$, *find* x.

Solution. $$\frac{\log x}{\log 3} = \frac{\log 3^4}{\log 3^2}$$

$$= \frac{4 \log 3}{2 \log 3} = 2$$

$\log x = 2 \log 3 = \log 3^2 = \log 9$

$x = 9$ **Ans.**

Example 31. *If* $a = \log_x yz$, $b = \log_y zx$, $c = \log_z xy$, *show that*

$$abc = a + b + c + 2$$

Solution.
$$\begin{aligned}
abc &= \log_x yz \cdot \log_y zx \cdot \log_z xy \\
&= (\log_x y + \log_x z)(\log_y z + \log_y x)(\log_z x + \log_z y) \\
&= (\log_x y \cdot \log_y z + \log_x y \cdot \log_y x + \log_x z \cdot \log_y z + \log_x z \log_y x)(\log_z x + \log_z y) \\
&= (\log_x z + 1 + \log_x z \log_y z + \log_y z)(\log_z x + \log_z y) \\
&= \log_x z \log_z x + \log_x z \log_z y + \log_z x + \log_z y + \log_x z \log_y z \log_z x \\
&\qquad + \log_x z \log_y z \log_z y + \log_y z \log_z x + \log_y z \log_z y \\
&= 1 + \log_x y + \log_z x + \log_z y + \log_y z + \log_x z + \log_y x + 1 \\
&= 2 + (\log_x y + \log_x z) + (\log_y z + \log_y x) + (\log_z x + \log_z y) \\
&= 2 + (\log_x yz + \log_y zx + \log_z xy) \\
&= 2 + a + b + c
\end{aligned}$$
Proved.

Example 32. $5^{2x} \cdot 7^{2x-1} = 175$

Solution.
$$\begin{aligned}
& \log 5^{2x} \cdot 7^{2x-1} = \log 175 \\
\Rightarrow\ & \log 5^{2x} + \log 7^{2x-1} = \log (5^2 \times 7) \\
\Rightarrow\ & 2x \log 5 + (2x - 1)\log 7 = 2 \log 5 + \log 7 \\
\Rightarrow\ & 2x \log 5 - 2 \log 5 + (2x - 1)\log 7 - \log 7 = 0 \\
\Rightarrow\ & (2x - 2) \log 5 + (2x - 2) \log 7 = 0 \\
\Rightarrow\ & (2x - 2) (\log 5 + \log 7) = 0 \\
\Rightarrow\ & (2x - 2) \log 35 = 0 \\
\therefore\ & 2x - 2 = 0 \quad \text{or} \quad x = 1.
\end{aligned}$$
Ans.

1.10 COMMON LOGARITHM

Logarithm of numbers to the base 10 are called *common logarithm*. In numerical calculations, the common logarithms are used. Whenever the base is not written, we shall assume the base to be 10.

$$\begin{array}{llll}
10^3 = 1000 & \Rightarrow & \log (1000) = 3 & \text{or} \quad \log 10^3 = 3 \\
10^2 = 100 & \Rightarrow & \log (100) = 2 & \text{or} \quad \log 10^2 = 2 \\
10^1 = 10 & \Rightarrow & \log (10) = 1 & \text{or} \quad \log 10^1 = 1 \\
10^0 = 1 & \Rightarrow & \log (1) = 0 & \text{or} \quad \log 10^0 = 0 \\
10^{-1} = 0.1 & \Rightarrow & \log (0.1) = -1 & \text{or} \quad \log 10^{-1} = -1 \\
10^{-2} = 0.01 & \Rightarrow & \log (0.01) = -2 & \text{or} \quad \log 10^{-2} = -2 \\
10^{-3} = 0.001 & \Rightarrow & \log (0.001) = -3 & \text{or} \quad \log 10^{-3} = -3
\end{array}$$

and so on.

We conclude that if n is an integer, then $\log_{10} 10^n = n$.

Logarithm to the base e are called natural or Naperian logarithms. Naperian logarithms are used in differential calculus, integral calculus, etc.

1.11 CHARACTERISTIC AND MANTISSA

We know

$$\begin{array}{lll}
10^3 = 1000 & \Rightarrow & \log (1000) = 3 \\
10^2 = 100 & \Rightarrow & \log (100) = 2
\end{array}$$

$10^1 = 10 \quad \Rightarrow \quad \log(10) = 1$

$10^0 = 1 \quad \Rightarrow \quad \log(1) = 0$

$10^{-1} = 0.1 \quad \Rightarrow \quad \log(0.1) = -1$

$10^{-2} = 0.01 \quad \Rightarrow \quad \log(0.01) = -2$

$10^{-3} = 0.001 \quad \Rightarrow \quad \log(0.001) = -3$

Logarithms of a number between 1000 and 100 lies between 3 and 2 (2 + a decimal).

Logarithm of a number between 100 and 10 lies between 2 and 1 (1 + a decimal), and so on.

Numbers	Logarithm
Between 1000 and 100	$2 < \log < 3$ or 2 + a decimal
Between 100 and 10	$1 < \log < 2$ or 1 + a decimal
Between 10 and 1	$0 < \log < 1$ or 0 + a decimal
Between 1 and 0.1	$-1 < \log < 0$ or –1 + a decimal
Between 0.1 and 0.01	$-2 < \log < -1$ or –2 + a decimal
Between 0.01 and 0.001	$-3 < \log < -2$ or –3 + a decimal

Thus logarithm of any number consists of two parts:

(*i*) Integral part (*ii*) Decimal part

Characteristic: The integral part of logarithm is called the *characteristic*. It may be negative or positive.

Mantissa: The decimal part of logarithm is known as the *mantissa*. It is always positive.

$$\log(300) = 2.3010$$

Characteristic Mantissa

Number	Characteristic
825	2
36	1
4	0
0.3	–1
0.05	–2
0.002	–3

Rule to Know the Characteristic

Characteristic of the numbers greater than 1.

Number of	Characteristic
3 digits	2
2 digits	1
1 digit	0

EXERCISE 1.8

Find the value of each of the following:

1. $\log_2 16\sqrt{8}$ **Ans.** $\frac{11}{2}$

2. $\log_{10} \sqrt[3]{100}$ **Ans.** $\frac{2}{3}$

3. $\log_5 \frac{\sqrt[4]{25}}{625}$ **Ans.** $-\frac{7}{2}$

Prove that the following:

4. $\log 12 = \log 3 + 2 \log 2$

5. $\log 360 = 3 \log 2 + 2 \log 3 + \log 5$

6. $3 \log 2 + \log 5 = \log 40$

7. $5 \log 3 + \log 9 = \log 2187$

8. $\log\left(\frac{9}{14}\right) + \log\left(\frac{35}{24}\right) - \log\left(\frac{15}{16}\right) = 0$

9. $\log\left(\frac{8}{49}\right) + \log\left(\frac{28}{3}\right) + \log\left(\frac{21}{16}\right) = \log 2$

10. $2\log\left(\frac{16}{15}\right) - \log\left(\frac{24}{25}\right) + \log\left(\frac{27}{32}\right) = 0$

11. If $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find $\log 24$. **Ans.** 1.3801

12. Simplify $\frac{\log_2 11}{\log_2 7} - \frac{\log_5 11}{\log_5 7}$ **Ans.** 0

13. $\frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1$

14. $\log\left(\frac{4}{5}\right) + \log\left(\frac{5}{7}\right) + \log\left(\frac{7}{4}\right) = 0$

15. $\log\left(\frac{a^2}{bc}\right) + \log\left(\frac{b^2}{ca}\right) + \log\left(\frac{c^2}{ab}\right) = 0$

16. $\log\left(\frac{8}{49}\right) + \log\left(\frac{28}{3}\right) + \log\left(\frac{21}{16}\right) = \log 2$

17. $2\log\left(\frac{16}{15}\right) - \log\left(\frac{24}{25}\right) + \log\left(\frac{27}{32}\right) = 0$

18. $\log\left(\frac{75}{16}\right) - 2\log\left(\frac{5}{6}\right) + \log\left(\frac{8}{27}\right) = \log 2$

19. $\log (3 + 3) \neq \log 3 + \log 3$

20. $\log (1 + 2 + 3) = \log 1 + \log 2 + \log 3$

21. $\log 2 + \log 4 > \log (2 + 4)$

22. $\log 6 - \log 3 < \log (6 - 3)$

23. If $x = \log_7 27$, $y = \log_5 7$, $z = \log_3 5$, show that $xyx = 3$.

24. If $a^2 + b^2 = 7ab$, Show that $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}[\log a + \log b]$.

25. If $x^2 + y^2 = 8\, xy$, show that $2 \log(x + y) = \log 5 + \log 2 + \log x + \log y$.

26. If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$, show that $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1$.

27. Prove that $x^{\log y - \log z} \cdot y^{\log z - \log x} \cdot z^{\log x - \log y} = 1$

28. Solve: $\log_3 2x + \log_3 7 = \log_3 5$ **Ans.** $x = \frac{5}{14}$