CHAPTER 1

MATHEMATICAL MODELS OF CONTROL SYSTEM

1.1 CONTROL SYSTEM

Control system theory evolved as an engineering discipline and due to universality of the principles involved, it is extended to various fields like economy, sociology, biology, medicine, etc. Control theory has played a vital role in the advance of engineering and science. The automatic control has become an integral part of modern manufacturing and industrial processes. For example, numerical control of machine tools in manufacturing industries, controlling pressure, temperature, humidity, viscosity and flow in process industry.

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a ***system.*** In a system when the output quantity is controlled by varying the input quantity, the system is called ***control system.*** The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

OPEN LOOP SYSTEM

Any physical system which does not automatically correct the variation in its output, is called an ***open loop system***, or control system in which the output quantity has no effect upon the input quantity are called open-loop control system. This means that the output is not fedback to the input for correction.

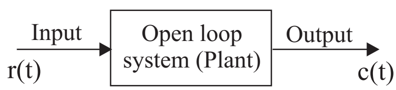


Fig 1.1 : *Open loop system.*

In open loop system the output can be varied by varying the input. But due to external disturbances the system output may change. When the output changes due to disturbances, it is not followed by changes in input to correct the output. In open loop systems the changes in output are corrected by changing the input manually.

CLOSED LOOP SYSTEM

Control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called ***closed loop systems.***

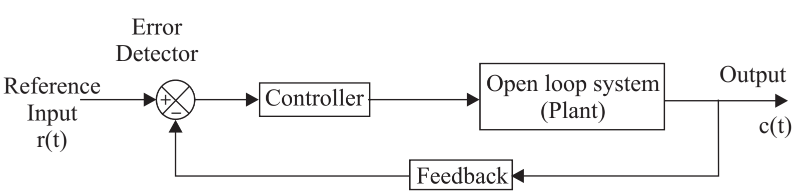


Fig 1.2 : *Closed loop system.*

The open loop system can be modified as closed loop system by providing a feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called ***automatic control system.*** The general block diagram of an automatic control system is shown in fig 1.2. It consists of an error detector, a controller, plant (open loop system) and feedback path elements.

The reference signal (or input signal) corresponds to desired output. The feedback path elements samples the output and converts it to a signal of same type as that of reference signal. The feedback signal is proportional to output signal and it is fed to the error detector. The error signal generated by the error detector is the difference between reference signal and feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output.

Advantages of open loop systems

1. The open loop systems are simple and economical.
2. The open loop systems are easier to construct.
3. Generally the open loop systems are stable.

Disadvantages of open loop systems

1. The open loop systems are inaccurate and unreliable.
2. The changes in the output due to external disturbances are not corrected automatically.

Advantages of closed loop systems

1. The closed loop systems are accurate.
2. The closed loop systems are accurate even in the presence of non-linearities.
3. The sensitivity of the systems may be made small to make the system more stable.
4. The closed loop systems are less affected by noise.

Disadvantages of closed loop systems

1. The closed loop systems are complex and costly.
2. The feedback in closed loop system may lead to oscillatory response.
3. The feedback reduces the overall gain of the system.
4. Stability is a major problem in closed loop system and more care is needed to design a stable closed loop system.

1.2 EXAMPLES OF CONTROL SYSTEMS

EXAMPLE 1 : TEMPERATURE CONTROL SYSTEM

OPEN LOOP SYSTEM

The electric furnace shown in fig 1.3. is an open loop system. The output in the system is the desired temperature. The temperature of the system is raised by heat generated by the heating element. The output temperature depends on the time during which the supply to heater remains ON.

The ON and OFF of the supply is governed by the time setting of the relay. The temperature is measured by a sensor, which gives an analog voltage corresponding to the temperature of the furnace. The analog signal is converted to digital signal by an Analog - to - Digital converter (A/D converter).

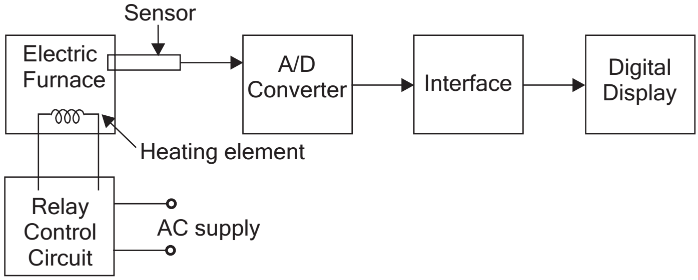


Fig 1.3 : *Open loop temperature control system.*

The digital signal is given to the digital display device to display the temperature. In this system if there is any change in output temperature then the time setting of the relay is not altered automatically.

CLOSED LOOP SYSTEM

The electric furnace shown in fig 1.4 is a closed loop system. The output of the system is the desired temperature and it depends on the time during which the supply to heater remains ON.

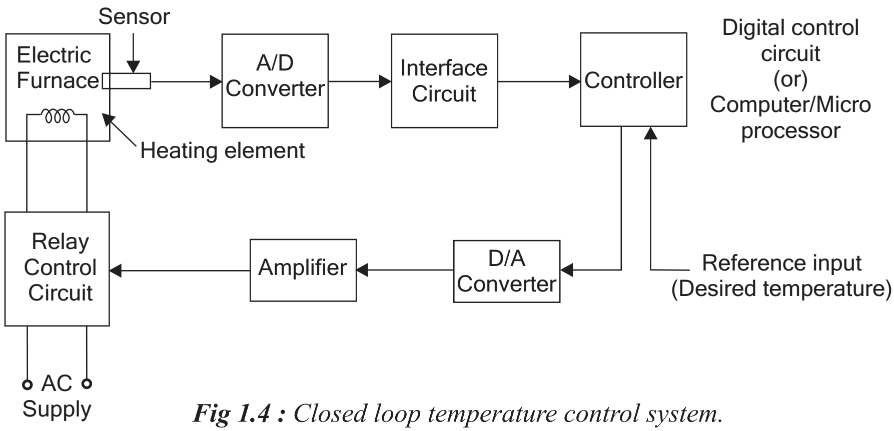


Fig 1.4 : *Closed loop temperature control system.*

The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer. The desired temperature is input to the system through keyboard or as a signal corresponding to desired temperature via ports. The actual temperature is sensed by sensor and converted to digital signal by the A/D converter. The computer reads the actual temperature and compares with desired temperature. If it finds any difference then it sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus the system automatically corrects any changes in output. Hence it is a closed loop system.

EXAMPLE 2 : TRAFFIC CONTROL SYSTEM

OPEN LOOP SYSTEM

Traffic control by means of traffic signals operated on a time basis constitutes an open-loop control system. The sequence of control signals are based on a time slot given for each signal. The time slots are decided based on a traffic study. The system will not measure the density of the traffic before giving the signals. Since the time slot does not changes according to traffic density, the system is open loop system.

CLOSED LOOP SYSTEM

Traffic control system can be made as a closed loop system if the time slots of the signals are decided based on the density of traffic. In closed loop traffic control system, the density of the traffic is measured on all the sides and the information is fed to a computer. The timings of the control signals are decided by the computer based on the density of traffic . Since the closed loop system dynamically changes the timings, the flow of vehicles will be better than open loop system.

EXAMPLE 3 : NUMERICAL CONTROL SYSTEM

OPEN LOOP SYSTEM

Numerical control is a method of controlling the motion of machine components using numbers. Here, the position of work head tool is controlled by the binary information contained in a disk.

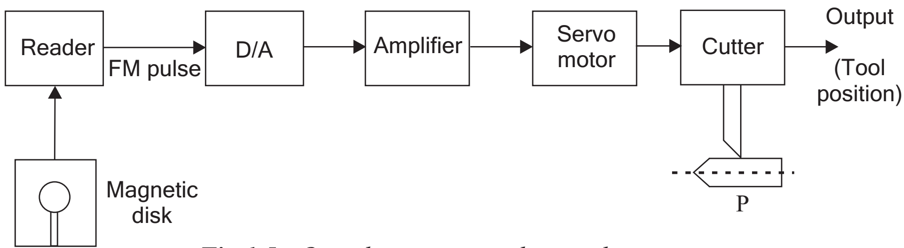


Fig 1.5 : *Open loop numerical control system.*

A magnetic disk is prepared in binary form representing the desired part P (P is the metal part to be machined). The tool will operate on the desired part P. To start the system, the disk is fed through the reader to the D/A converter. The D/A converter converts the FM(frequency modulated) output of the reader to a analog signal. It is amplified and fed to servometer which positions the cutter on the desired part P. The position of the cutter head is controlled by the angular motion of the servometer. This is an open loop system since no feedback path exists between the output and input. The system positions the tool for a given input command. Any deviation in the desired position is not checked and corrected automatically.

CLOSED LOOP SYSTEM

A magnetic disk is prepared in binary form representing the desired part P (P is the metal part to be machined). To start the system, the disk is loaded in the reader. The controller compares the frequency modulated input pulse signal with the feedback pulse signal. The controller is a computer or microprocessor system. The controller carries out mathematical operations on the difference in the pulse signals and generates an error signal. The D/A converter converts the controller output pulse (error signal) into an analog signal . The amplified analog signal rotates the servomotor to position the tool on the job. The position of the cutterhead is controlled according to the input of the servomotor.

The transducer attached to the cutterhead converts the motion into an electrical signal. The analog electrical signal is converted to the digital pulse signal by the A/D converter. Then this signal is compared with the input pulse signal. If there is any difference between these two, the controller sends a signal to the servomotor to reduce it. Thus the system automatically corrects any deviation in the desired output tool position. An advantage of numerical control is that complex parts can be produced with uniform tolerances at the maximum milling speed.

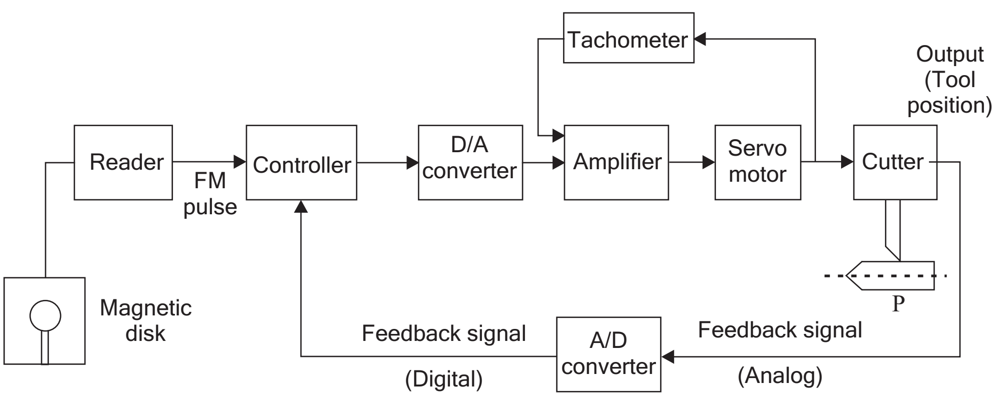


Fig 1.6 : *Closed loop numerical control system.*

EXAMPLE 4 : POSITION CONTROL SYSTEM USING SERVOMOTOR

The position control system shown in fig 1.7 is a closed loop system. The system consists of a servomotor powered by a generator. The load whose position has to be controlled is connected to motor shaft through gear wheels. Potentiometers are used to convert the mechanical motion to electrical signals. The desired load position (θ_R) is set on the input potentiometer and the actual load position (θ_c) is fed to feedback potentiometer. The difference between the two angular positions generates an error signal, which is amplified and fed to generator field circuit. The induced emf of the generator drives the motor. The rotation of the motor stops when the error signal is zero, i.e. when the desired load position is reached.

This type of control systems are called servomechanisms .The servo or servomechanisms are feedback control systems in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

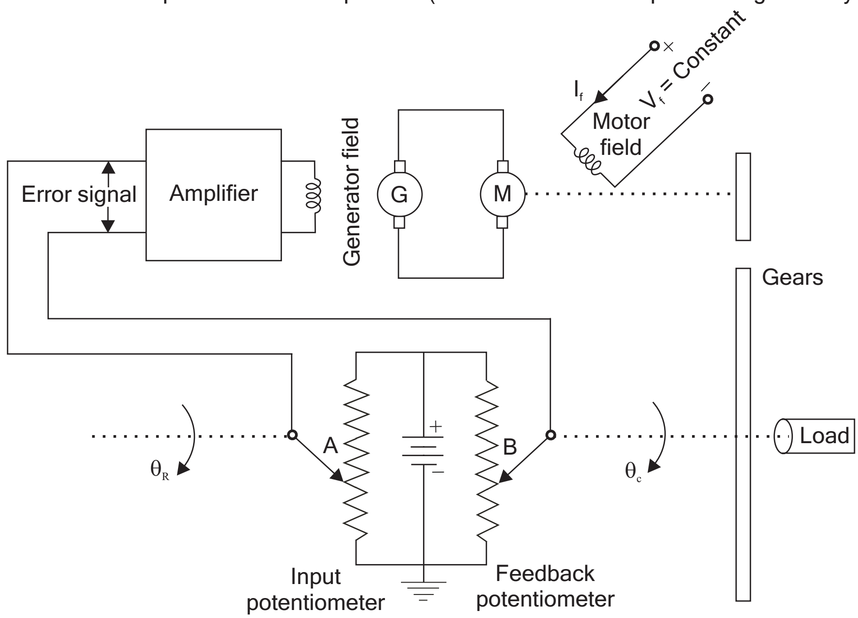


Fig 1.7 : *A position control system (servomechanism).*

1.3 MATHEMATICAL MODELS OF CONTROL SYSTEMS

A ***control system*** is a collection of physical objects (components) connected together to serve an objective. The input output relations of various physical components of a system are governed by ***differential equations.*** The mathematical model of a control system constitutes a set of differential equations. The response or output of the system can be studied by solving the differential equations for various input conditions.

The mathematical model of a system is linear if it obeys the principle of superposition and homogenity. This principle implies that if a system model has responses $y_1(t)$ and $y_2(t)$ to any inputs $x_1(t)$ and $x_2(t)$ respectively, then the system response to the linear combination of these inputs $a_1x_1(t) + a_2x_2(t)$ is given by linear combination of the individual outputs $a_1y_1(t) + a_2y_2(t)$, where a_1 and a_2 are constants.

The principle of superposition can be explained diagrammatically as shown in fig. 1.8.

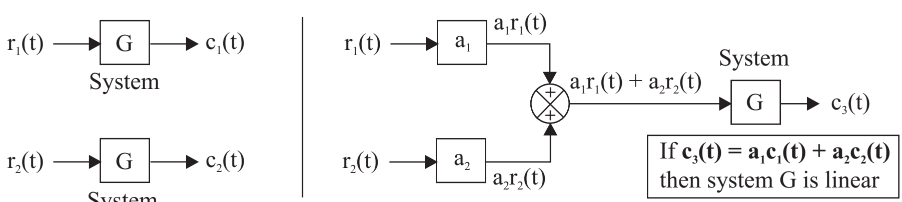


Fig 1.8 : *Principle of linearity and superposition.*

A mathematical model will be linear if the differential equations describing the system has constant coefficients (or the coefficients may be functions of independent variables). If the coefficients of the differential equation describing the system are constants then the model is ***linear time invariant.*** If the coefficients of differential equations governing the system are functions of time then the model is ***linear time varying.***

The differential equations of a linear time invariant system can be reshaped into different form for the convenience of analysis. One such model for single input and single output system analysis is transfer function of the system. The ***transfer function*** of a system is defined as the ratio of Laplace transform of output to the Laplace transform of input with zero initial conditions.

$$\boxed{\text{Transfer function} = \left.\frac{\text{Laplace Transform of output}}{\text{Laplace Transform of input}}\right|_{\text{with zero initial condition}}} \qquad .....(1.1)$$

The transfer function can be obtained by taking Laplace transform of the differential equations governing the system with zero initial conditions and rearranging the resulting algebraic equations to get the ratio of output to input.

1.4 MECHANICAL TRANSLATIONAL SYSTEMS

The model of mechanical translational systems can be obtained by using three basic elements ***mass, spring and dash-pot.*** These three elements represents three essential phenomena which occur in various ways in mechanical systems.

The weight of the mechanical system is represented by the element ***mass*** and it is assumed to be concentrated at the center of the body. The elastic deformation of the body can be represented by a ***spring.*** The friction existing in rotating mechanical system can be represented by the ***dash-pot.*** The dash-pot is a piston moving inside a cylinder filled with viscous fluid.

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body are governed by ***Newton's second law of motion.*** For translational systems it states that the sum of forces acting on a body is zero. (or Newton's second law states that the sum of applied forces is equal to the sum of opposing forces on a body).

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

x = Displacement, m

$v = \dfrac{dx}{dt}$ = Velocity, m/sec

$a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$ = Acceleration, m/sec^2

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N

f_k = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash - pot), N

M = Mass, kg

K = Stiffness of spring, N/m

B = Viscous friction co-efficient, N-sec/m

Note : *Lower case letters are functions of time*

FORCE BALANCE EQUATIONS OF IDEALIZED ELEMENTS

Consider an ideal mass element shown in fig 1.9 which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body.

Let, f = Applied force

f_m = Opposing force due to mass

Here, $f_m \propto \frac{d^2x}{dt^2}$ or $f_m = M\frac{d^2x}{dt^2}$

By Newton's second law, $\boxed{f = f_m = M\,\frac{d^2x}{dt^2}}$ .....(1.2)

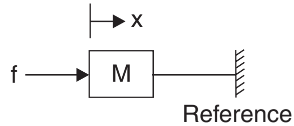


***Fig 1.9 :** Ideal mass element.*

Consider an ideal frictional element dashpot shown in fig 1.10 which has negligible mass and elasticity. Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body.

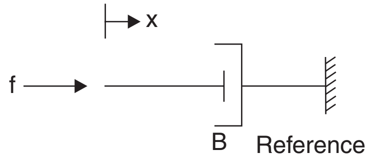


***Fig 1.10 :** Ideal dashpot with one end fixed to reference.*

Let, f = Applied force

f_b = Opposing force due to friction

Here, $f_b \propto \frac{dx}{dt}$ or $f_b = B\,\frac{dx}{dt}$

By Newton's second law, $\boxed{f = f_b = B\,\frac{dx}{dt}}$ .....(1.3)

When the dashpot has displacement at both ends as shown in fig 1.11, the opposing force is proportional to difference between velocity at both ends.

$f_b \propto \frac{d}{dt}(x_1 - x_2)$ or $f_b = B\,\frac{d}{dt}(x_1 - x_2)$

$\therefore \boxed{f = f_b = B\,\frac{d}{dt}(x_1 - x_2)}$ .....(1.4)

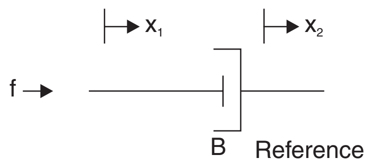


***Fig 1.11 :** Ideal dashpot with displacement at both ends.*

Consider an ideal elastic element spring shown in fig 1.12, which has negligible mass and friction. Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

Let, f = Applied force

f_k = Opposing force due to elasticity

Here $f_k \propto x$ or $f_k = K\,x$

By Newton's second law, $\boxed{f = f_k = Kx}$ .....(1.5)

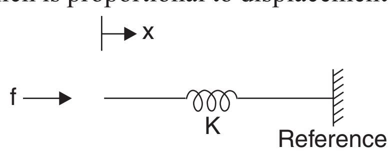


***Fig 1.12 :** Ideal spring with one end fixed to reference.*

When the spring has displacement at both ends as shown in fig 1.13 the opposing force is proportional to difference between displacement at both ends.

$f_k \propto (x_1 - x_2)$ or $f_k = K(x_1 - x_2)$

$\therefore \boxed{f = f_k = K(x_1 - x_2)}$ .....(1.6)

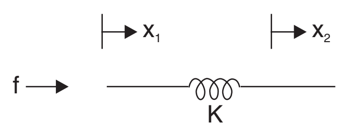


***Fig 1.13 :** Ideal spring with displacement at both ends.*

Guidelines to determine the Transfer Function of Mechanical Translational System

1. In mechanical translational system, the differential equations governing the system are obtained by writing force balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements in the system. In some cases the nodes may be without mass element.

2. The linear displacement of the masses (nodes) are assumed as x_1, x_2, x_3, etc., and assign a displacement to each mass(node).The first derivative of the displacement is velocity and the second derivative of the displacement is acceleration.

3. Draw the free body diagrams of the system. The free body diagram is obtained by drawing each mass separately and then marking all the forces acting on that mass (node). Always the opposing force acts in a direction opposite to applied force. The mass has to move in the direction of the applied force. Hence the displacement, velocity and acceleration of the mass will be in the direction of the applied force. If there is no applied force then the displacement, velocity and acceleration of the mass will be in a direction opposite to that of opposing force.

4. For each free body diagram, write one differential equation by equating the sum of applied forces to the sum of opposing forces.

5. Take Laplace transform of differential equations to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the ratio between output variable and input variable. This ratio is the transfer function of the system.

Note : *Laplace transform of* $x(t) = \mathcal{L}\{x(t)\} = X(s)$

Laplace transform of $\dfrac{dx(t)}{dt} = \mathcal{L}\left\{\dfrac{d}{dt}x(t)\right\} = s\,X(s)$ *(with zero initial conditions)*

Laplace transform of $\dfrac{d^2x(t)}{dt^2} = \mathcal{L}\left\{\dfrac{d^2}{dt^2}x(t)\right\} = s^2\,X(s)$ *(with zero initial conditions)*

EXAMPLE 1.1

Write the differential equations governing the mechanical system shown in fig 1. and determine the transfer function.

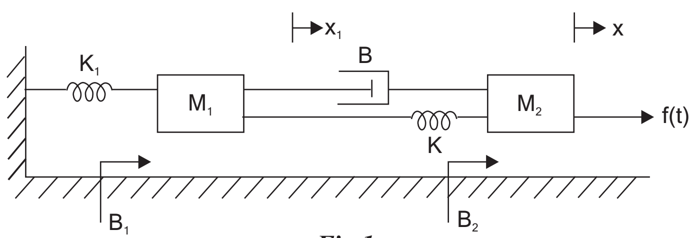


Fig 1.

SOLUTION

In the given system, applied force ‘f(t)’ is the input and displacement ‘x’ is the output.

Let, Laplace transform of f(t) $= \mathcal{L}\{f(t)\} = F(s)$

Laplace transform of x $= \mathcal{L}\{x\} = X(s)$

Laplace transform of $x_1 = \mathcal{L}\{x_1\} = X_1(s)$

Hence the required transfer function is $\frac{X(s)}{F(s)}$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are given by force balance equations at these nodes.

Let the displacement of mass M_1 be x_1. The free body diagram of mass M_1 is shown in fig 2. The opposing forces acting on mass M_1 are marked as f_{m1}, f_{b1}, f_b, f_{k1} and f_k.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt} \quad ; \quad f_{k1} = K_1 x_1$$

$$f_b = B \frac{d}{dt}(x_1 - x) \quad ; \quad f_k = K(x_1 - x)$$

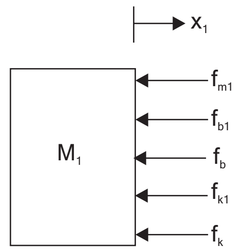


***Fig 2 :** Free body diagram of mass M_1 (node 1).*

By Newton's second law,

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$\therefore \; M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B \frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs[X_1(s) - X(s)] + K_1 X_1(s) + K[X_1(s) - X(s)] = 0$$

$$X_1(s)\,[M_1 s^2 + (B_1 + B)s + (K_1 + K)] - X(s)\,[Bs + K] = 0$$

$$X_1(s)\,[M_1 s^2 + (B_1 + B)s + (K_1 + K)] = X(s)\,[Bs + K]$$

$$\therefore \; X_1(s) = X(s) \frac{Bs + K}{M_1 s^2 + (B_1 + B)s + (K_1 + K)} \qquad \text{......(1)}$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on M_2 are marked as f_{m2}, f_{b2}, f_b and f_k.

$$f_{m2} = M_2 \frac{d^2x}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx}{dt}$$

$$f_{m2} = B \frac{dx}{dt}(x - x_1) \quad ; \quad f_k = K(x - x_1)$$

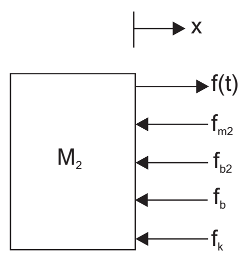


***Fig 3 :** Free body diagram of mass M_2 (node 2).*

By Newton's second law,

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B \frac{d}{dt}(x - x_1) + K(x - x_1) = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_2 s^2 X(s) + B_2 s X(s) + Bs[X(s) - X_1(s)] + K[X(s) - X_1(s)] = F(s)$$

$$X(s)[M_2 s^2 + (B_2 + B)s + K] - X_1(s)[Bs + K] = F(s) \qquad \text{.....(2)}$$

Substituting for $X_1(s)$ from equation (1) in equation (2) we get,

$$X(s)\,[M_2 s^2 + (B_2 + B)s + K] - X(s) \frac{(Bs + K)^2}{M_1 s^2 + (B_1 + B)s + (K_1 + K)} = F(s)$$

$$X(s)\left[\frac{[M_2s^2+(B_2+B)s+K][M_1s^2+(B_1+B)s+(K_1+K)]-(Bs+K)^2}{M_1s^2+(B_1+B)s+(K_1+K)}\right]=F(s)$$

$$\therefore \frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)][M_2s^2+(B_2+B)s+K]-(Bs+K)^2}$$

RESULT

The differential equations governing the system are,

1. $M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B\frac{d}{dt}(x_1-x)+K_1x_1+K(x_1-x)=0$

2. $M_2\frac{d^2x}{dt^2}+B_2\frac{dx}{dt}+B\frac{d}{dt}(x-x_1)+K(x-x_1)=f(t)$

The transfer function of the system is,

$$\frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)][M_2s^2+(B_2+B)s+K]-(Bs+K)^2}$$

EXAMPLE 1.2

Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown in fig 1.

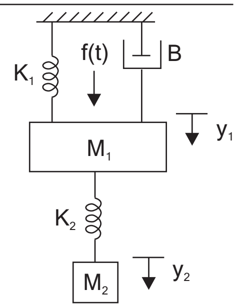


Fig 1.

SOLUTION

Let, Laplace transform of $f(t) = \mathcal{L}\{f(t)\} = F(s)$

Laplace transform of $y_1 = \mathcal{L}\{y_1\} = Y_1(s)$

Laplace transform of $y_2 = \mathcal{L}\{y_2\} = Y_2(s)$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are the force balance equations at these nodes.

The free body diagram of mass M_1 is shown in fig 2.

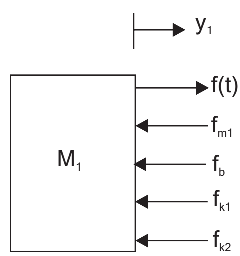


Fig 2.

The opposing forces are marked as f_{m1}, f_b, f_{k1} and f_{k2}

$$f_{m1}=M_1\frac{d^2y_1}{dt^2}\ ;\ f_b=B\frac{dy_1}{dt}\ ;\ f_{k1}=K_1y_1\ ;\ f_{k2}=K_2(y_1-y_2)$$

By Newton's second law, $f_{m1}+f_b+f_{k1}+f_{k2}=f(t)$

$$\therefore M_1\frac{d^2y_1}{dt^2}+B\frac{dy_1}{dt}+K_1y_1+K_2(y_1-y_2)=f(t) \quad .....(1)$$

On taking Laplace transform of equation (1) with zero initial condition we get,

$$M_1s^2Y_1(s)+BsY_1(s)+K_1Y_1(s)+K_2[Y_1(s)-Y_2(s)]=F(s)$$

$$Y_1(s)[M_1s^2+Bs+(K_1+K_2)]-Y_2(s)K_2=F(s) \quad .....(2)$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on M_2 are f_{m2} and f_{k2}.

$$f_{m2}=M_2\frac{d^2y_2}{dt^2}\ ;\ f_{k2}=K_2(y_2-y_1)$$

By Newton's second law, $f_{m2}+f_{k2}=0$

$$\therefore\ M_2\frac{d^2y_2}{dt^2}+K_2(y_2-y_1)=0$$

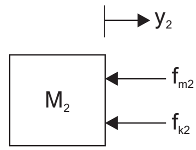


Fig 3.

On taking Laplace transform of above equation we get,

$$M_2s^2Y_2(s)+K_2[Y_2(s)-Y_1(s)]=0$$

$$Y_2(s)\,[M_2s^2+K_2]-Y_1(s)\,K_2=0$$

$$\therefore\ Y_1(s)=Y_2(s)\,\frac{M_2s^2+K_2}{K_2} \quad .....(3)$$

Substituting for $Y_1(s)$ from equation (3) in equation (2) we get,

$$Y_2(s)\left[\frac{M_2s^2+K_2}{K_2}\right][M_1s^2+Bs+(K_1+K_2)]-Y_2(s)\,K_2=F(s)$$

$$Y_2(s)\left[\frac{(M_2s^2+K_2)\,[M_1s^2+Bs+(K_1+K_2)]-K_2^2}{K_2}\right]=F(s)$$

$$\therefore\ \frac{Y_2(s)}{F(s)}=\left[\frac{K_2}{[M_1s^2+Bs+(K_1+K_2)]\,(M_2s^2+K_2)-K_2^2}\right]$$

RESULT

The differential equations governing the system are,

1. $M_1\dfrac{d^2y_1}{dt^2}+B\dfrac{dy_1}{dt}+K_1y_1+K_2(y_1-y_2)=f(t)$

2. $M_2\dfrac{d^2y_2}{dt^2}+K_2(y_2-y_1)=0$

The transfer function of the system is,

$$\frac{Y_2(s)}{F(s)}=\frac{K_2}{[M_1s^2+Bs+(K_1+K_2)]\,[M_2s^2+K_2]-K_2^2}$$

EXAMPLE 1.3

Determine the transfer function, $\dfrac{X_1(s)}{F(s)}$ and $\dfrac{X_2(s)}{F(s)}$ for the system shown in fig 1.

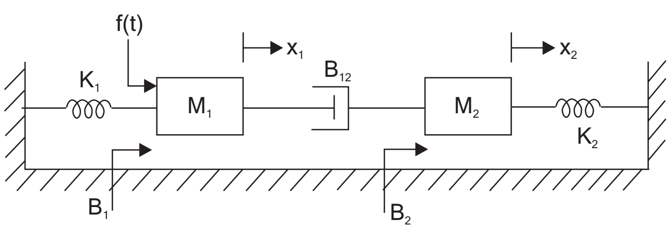


Fig 1.

SOLUTION

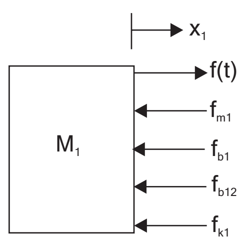


Fig 2.

Let, Laplace transform of f(t) = $\mathcal{L}\{f(t)\}$ = F(s)

Laplace transform of x_1 = $\mathcal{L}\{x_1\}$ = $X_1(s)$

Laplace transform of x_2 = $\mathcal{L}\{x_2\}$ = $X_2(s)$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are the force balance equations at these nodes. The free body diagram of mass M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.

$$f_{m1} = M_1\frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1\frac{dx_1}{dt} \quad ; \quad f_{b12} = B_{12}\frac{d}{dt}(x_1 - x_2) \quad ; \quad f_{k1} = K_1x_1$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B_{12}\frac{d(x_1 - x_2)}{dt} + K_1x_1 = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1s^2X_1(s) + B_1sX_1(s) + B_{12}s[X_1(s) - X_2(s)] + K_1X_1(s) = F(s)$$

$$X_1(s)[M_1s^2 + (B_1 + B_{12})s + K_1] - B_{12}sX_2(s) = F(s) \qquad .....(1)$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b12} and f_{k2}.

$$f_{m2} = M_2\frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2\frac{dx_2}{dt}$$

$$f_{b12} = B_{12}\frac{d}{dt}(x_2 - x_1) \quad ; \quad f_{k2} = K_2x_2$$

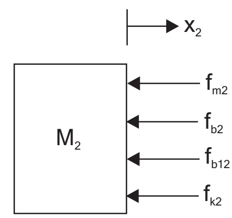


Fig 3.

By Newton's second law, $f_{m2} + f_{b2} + f_{b12} + f_{k2} = 0$

$$M_2\frac{d^2x_2}{dt^2} + B_2\frac{dx_2}{dt} + B_{12}\frac{d(x_2 - x_1)}{dt} + K_2x_2 = f(t) \qquad .....(2)$$

On taking Laplace transform of equation (2) with zero initial conditions we get,

$$M_2s^2X_2(s) + B_2sX_2(s) + B_{12}s[X_2(s) - X_1(s)] + K_2X_2(s) = 0$$

$$X_2(s)[M_2s^2 + (B_2 + B_{12})s + K_2] - B_{12}sX_1(s) = 0$$

$$X_2(s)[M_2s^2 + (B_2 + B_{12})s + K_2] = B_{12}sX_1(s)$$

$$X_2(s) = \frac{B_{12}sX_1(s)}{[M_2s^2 + (B_2 + B_{12})s + K_2]} \qquad .....(3)$$

Substituting for $X_2(s)$ from equation (3) in equation (1) we get,

$$X_1(s)\,[M_1s^2 + (B_1 + B_{12})s + K_1] - \frac{(B_{12}s)^2X_1(s)}{M_2s^2 + (B_2 + B_{12})s + K_2} = F(s)$$

$$\frac{X_1(s)\left[\,[M_1s^2 + (B_1 + B_{12})s + K_1]\,[M_2s^2 + (B_2 + B_{12})s + K_2] - (B_{12}s)^2\right]}{M_2s^2 + (B_2 + B_{12})s + K_2} = F(s)$$

$$\therefore\ \frac{X_1(s)}{F(s)} = \frac{M_2s^2 + (B_2 + B_{12})s + K_2}{[M_1s^2 + (B_1 + B_{12})s + K_1]\,[M_2s^2 + (B_2 + B_{12})s + K_2] - (B_{12}s)^2}$$

From equation (3) we get,

$$X_1(s) = \frac{[M_2s^2 + (B_2 + B_{12})s + K_2]X_2(s)}{B_{12}s} \qquad .....(4)$$

Substituting for $X_1(s)$ from equation (4) in equation (1) we get,

$$\frac{X_2(s)\,[M_2s^2 + (B_2 + B_{12})s + K_2]}{B_{12}s}\,[M_1s^2 + (B_1 + B_{12})s + K_1] - B_{12}sX_2(s) = F(s)$$

$$X_2(s)\left[\frac{[M_2s^2+(B_2+B_{12})s+K_2][M_1s^2+(B_1+B_{12})s+K_1]-(B_{12}s)^2}{B_{12}s}\right] = F(s)$$

$$\therefore\ \frac{X_2(s)}{F(s)} = \frac{B_{12}s}{[M_2s^2+(B_2+B_{12})s+K_2][M_1s^2+(B_1+B_{12})s+K_1]-(B_{12}s)^2}$$

RESULT

The differential equations governing the system are,

1. $M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B_{12}\frac{d(x_1-x_2)}{dt}+K_1x_1 = f(t)$

2. $M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+B_{12}\frac{d(x_2-x_1)}{dt}+K_2x_2 = 0$

The transfer functions of the system are,

1. $\frac{X_1(s)}{F(s)}=\frac{M_2s^2+(B_2+B_{12})s+K_2}{[M_1s^2+(B_1+B_{12})s+K_1][M_2s^2+(B_2+B_{12})s+K_2]-(B_{12}s)^2}$

2. $\frac{X_2(s)}{F(s)}=\frac{B_{12}s}{[M_2s^2+(B_2+B_{12})s+K_2][M_1s^2+(B_1+B_{12})s+K_1]-(B_{12}s)^2}$

EXAMPLE 1.4

Write the equations of motion in s-domain for the system shown in fig 1. Determine the transfer function of the system.

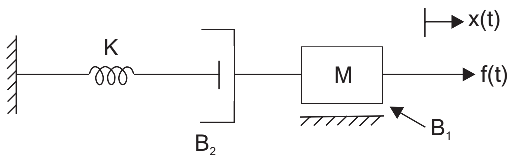


Fig 1.

SOLUTION

Let, Laplace transform of $x(t) = \mathcal{L}\{x(t)\} = X(s)$

Laplace transform of $f(t) = \mathcal{L}\{f(t)\} = F(s)$

Let x_1 be the displacement at the meeting point of spring and dashpot. Laplace transform of x_1 is $X_1(s)$.

The system has two nodes and they are mass M and the meeting point of spring and dashpot. The differential equations governing the system are the force balance equations at these nodes. The equations of motion in the s-domain are obtained by taking Laplace transform of the differential equations.

The free body diagram of mass M is shown in fig 2. The opposing forces are marked as f_m, f_{b1} and f_{b2}.

$$f_m = M\frac{d^2x}{dt^2} \quad ; \quad f_{b1} = B_1\frac{dx}{dt} \quad ; \quad f_{b2} = B_2\frac{d}{dt}(x-x_1)$$

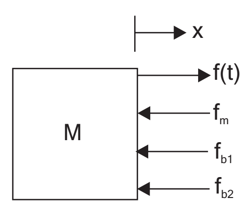


Fig 2.

By Newton's second law the force balance equation is,

$$f_m + f_{b1} + f_{b2} = f(t)$$

$$\therefore\ M\frac{d^2x}{dt^2}+B_1\frac{dx}{dt}+B_2\frac{d}{dt}(x-x_1) = f(t)$$

On taking Laplace transform of the above equation we get,

$$Ms^2X(s) + B_1sX(s) + B_2s[X(s) - X_1(s)] = F(s)$$

$$[Ms^2 + (B_1 + B_2)s]\,X(s) - B_2\,sX_1(s) = F(s) \qquad .....(1)$$

The free body diagram at the meeting point of spring and dashpot is shown in fig 3. The opposing forces are marked as f_k and f_{b2}.

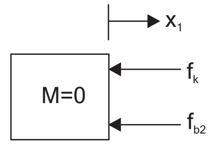


Fig 3.

$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x) \quad ; \quad f_k = Kx_1$$

By Newton's second law, $f_{b2} + f_k = 0$

$$\therefore\ B_2 \frac{d}{dt}(x_1 - x) + Kx_1 = 0$$

On taking Laplace transform of the above equation we get,

$$B_2 s[X_1(s) - X(s)] + K\, X_1(s) = 0$$

$$(B_2 s + K)\, X_1(s) - B_2\, sX(s) = 0$$

$$\therefore\ X_1(s) = \frac{B_2 s}{B_2 s + K} X(s) \qquad .....(2)$$

Substituting for $X_1(s)$ from equation (2) in equation (1) we get,

$$[\,Ms^2 + (B_1 + B_2)s\,]\, X(s) - B_2 s\left[\frac{B_2 s}{B_2 s + K}\right] X(s) = F(s)$$

$$X(s)\, \frac{[\,Ms^2 + (B_1 + B_2)s\,](B_2 s + K) - (B_2 s)^2]}{B_2 s + K} = F(s)$$

$$\therefore\ \frac{X(s)}{F(s)} = \frac{B_2 s + K}{[\,Ms^2 + (B_1 + B_2)s\,](B_2 s + K) - (B_2 s)^2]}$$

RESULT

The differential equations governing the system are,

1. $M \frac{d^2x}{dt^2} + B_1 \frac{dx}{dt} + B_2 \frac{d}{dt}(x - x_1) = f(t)$

2. $B_2 \frac{d}{dt^2}(x_1 - x) + K\, x_1 = 0$

The equations of motion in s-domain are,

1. $[M\, s^2 + (B_1 + B_2)s]\, X(s) - B_2\, sX_1(s) = F(s)$

2. $(B_2 s + K)\, X_1(s) - B_2\, sX(s) = 0$

The transfer function of the system is,

$$\frac{X(s)}{F(s)} = \frac{B_2 s + K}{[M\, s^2 + (B_1 + B_2)s]\,(B_2 s + K) - (B_2 s)^2}$$

1.5 MECHANICAL ROTATIONAL SYSTEMS

The model of rotational mechanical systems can be obtained by using three elements, ***moment of inertia*** [J] of mass, ***dash-pot*** with rotational frictional coefficient [B] and ***torsional spring*** with stiffness [K].

The weight of the rotational mechanical system is represented by the moment of inertia of the mass. The moment of inertia of the system or body is considered to be concentrated at the centre of gravity of the body. The elastic deformation of the body can be represented by a spring (torsional spring). The friction existing in rotational mechanical system can be represented by the dash-pot. The dash-pot is a piston rotating inside a cylinder filled with viscous fluid.

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torques acting on a rotational mechanical body are governed by ***Newton's second law of motion*** for rotational systems. It states that the sum of torques acting on a body is zero (or Newton's law states that the sum of applied torques is equal to the sum of opposing torques on a body).

LIST OF SYMBOLS USED IN MECHANICAL ROTATIONAL SYSTEM

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt}$ = Angular acceleration, rad/sec^2

T = Applied torque, N-m

J = Moment of inertia, Kg-m^2/rad

B = Rotational frictional coefficient, N-m/(rad/sec)

K = Stiffness of the spring, N-m/rad

TORQUE BALANCE EQUATIONS OF IDEALISED ELEMENTS

Consider an ideal mass element shown in fig 1.14 which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.

Let, T = Applied torque.

T_j = Opposing torque due to moment of inertia of the body.

Here $T_j \propto \frac{d^2\theta}{dt^2}$ or $T_j = J\frac{d^2\theta}{dt^2}$

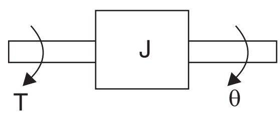


Fig 1.14 : *Ideal rotational mass element.*

By Newton's second law,

$$\boxed{T = T_j = J\frac{d^2\theta}{dt^2}} \quad .....(1.7)$$

Consider an ideal frictional element dash pot shown in fig 1.15 which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dash pot will offer an opposing torque which is proportional to the angular velocity of the body.

Let, T = Applied torque.

T_b = Opposing torque due to friction.

$T_b \propto \frac{d\theta}{dt^2}$ or $T_b = B\,\frac{d\theta}{dt}$

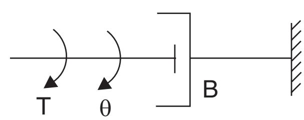


Fig 1.15 : *Ideal rotational dash-pot with one end fixed to reference.*

By Newton's second law, $$\boxed{T = T_b = B\,\frac{d\theta}{dt}} \quad .....(1.8)$$

When the dash pot has angular displacement at both ends as shown in fig 1.16, the opposing torque is proportional to the difference between angular velocity at both ends.

$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2)$ or $T_b = B\,\frac{d}{dt}(\theta_1 - \theta_2)$

$$\therefore \boxed{T = T_b = B\,\frac{d}{dt}(\theta_1 - \theta_2)} \quad .....(1.9)$$

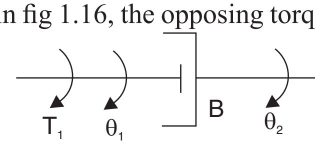


Fig 1.16 : *Ideal dash-pot with angular displacement at both ends.*

Consider an ideal elastic element, torsional spring as shown in fig 1.17, which has negligible moment of inertia and friction. Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.

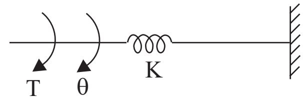


***Fig 1.17 :** Ideal spring with one end fixed to reference.*

Let, T = Applied torque.

T_k = Opposing torque due to elasticity.

$T_k \propto \theta$ or $T_k = K\theta$

By Newton's second law, $\boxed{T = T_k = K\theta}$ .....(1.10)

When the spring has angular displacement at both ends as shown in fig 1.18 the opposing torque is proportional to difference between angular displacement at both ends.

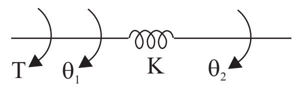


***Fig 1.18 :** Ideal spring with angular displacement at both ends.*

$T_k \propto (\theta_1 - \theta_2)$ or $T_k = K(\theta_1 - \theta_2)$

$\therefore \boxed{T = T_k = K(\theta_1 - \theta_2)}$ .....(1.11)

Guidelines to determine the Transfer Function of Mechanical Rotational System

1. In mechanical rotational system, the differential equations governing the system are obtained by writing torque balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements with moment of inertia in the system. In some cases the nodes may be without mass element.
2. The angular displacement of the moment of inertia of the masses (nodes) are assumed as θ_1, θ_2, θ_3, etc., and assign a displacement to each mass (node). The first derivative of angular displacement is angular velocity and the second derivative of the angular displacement is angular acceleration.
3. Draw the free body diagrams of the system. The free body diagram is obtained by drawing each moment of inertia of mass separately and then marking all the torques acting on that body. Always the opposing torques acts in a direction opposite to applied torque.
4. The mass has to rotate in the direction of the applied torque. Hence the angular displacement, velocity and acceleration of the mass will be in the direction of the applied torque. If there is no applied torque then the angular displacement, velocity and acceleration of the mass is in a direction opposite to that of opposing torque.
5. For each free body diagram write one differential equation by equating the sum of applied torques to the sum of opposing torques.
6. Take Laplace transform of differential equation to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the relation between output variable and input variable. This ratio is the transfer function of the system.

Note :

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\dfrac{d\theta}{dt} = \mathcal{L}\left\{\dfrac{d\theta}{dt}\right\} = s\,\theta(s)$ *(with zero initial conditions)*

Laplace transform of $\dfrac{d^2\theta}{dt^2} = \mathcal{L}\left\{\dfrac{d^2\theta}{dt^2}\right\} = s\,\theta(s)$ *(with zero initial conditions)*

EXAMPLE 1.5

Write the differential equations governing the mechanical rotational system shown in fig 1. Obtain the transfer function of the system.

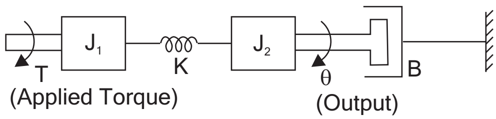


Fig 1.

SOLUTION

In the given system, applied torque T is the input and angular displacement θ is the output.

Let, Laplace transform of T $= \mathcal{L}\{T\} = T(s)$

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\theta_1 = \mathcal{L}\{\theta_1\} = \theta_1(s)$

Hence the required transfer function is $\dfrac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The free body diagram of J_1 is shown in fig 2. The opposing torques acting on J_1 are marked as T_{j1} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_k = K(\theta_1 - \theta)$$

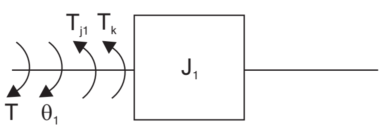


Fig 2 : *Free body diagram of mass with moment of inertia J_1.*

By Newton's second law, $T_{j1} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T \qquad .....(1)$$

On taking Laplace transform of equation (1) with zero initial conditions we get,

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$(J_1 s^2 + K)\theta_1(s) - K\theta(s) = T(s) \qquad .....(2)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig 3. The opposing torques acting on J_2 are marked as T_{j2}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_b = B\frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

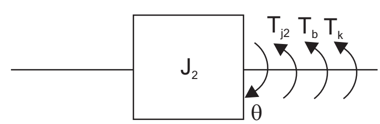


Fig 3 : *Free body diagram of mass with moment of inertia J_2.*

By Newton's second law, $T_{j2} + T_b + T_k = 0$

$$\therefore \; J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2 s^2 \theta(s) + B\, s\, \theta(s) + K\, \theta(s) - K\theta_1(s) = 0$$

$$(J_2s^2 + Bs + K)\,\theta(s) - K\theta_1(s) = 0$$

$$\theta_1(s) = \frac{(J_2s^2 + Bs + K)}{K}\,\theta(s) \qquad .....(3)$$

Substituting for $\theta_1(s)$ from equation (3) in equation (2) we get,

$$(J_1s^2 + K)\,\frac{(J_2s^2 + Bs + K)}{K}\,\theta(s) - K\theta(s) = T(s)$$

$$\left[\frac{(J_1s^2 + K)(J_2s^2 + Bs + K) - K^2}{K}\right]\theta(s) = T(s)$$

$$\therefore\ \frac{\theta(s)}{T(s)} = \frac{K}{(J_1s^2 + K)(J_2s^2 + Bs + K) - K^2}$$

RESULT

The differential equations governing the system are,

1. $J_1\dfrac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T$

2. $J_2\dfrac{d^2\theta}{dt^2} + B\dfrac{d\theta}{dt} + K\theta - K\theta_1 = 0$

The transfer function of the system is,

$$\frac{\theta(s)}{T(s)} = \frac{K}{(J_1s^2 + K)(J_2s^2 + Bs + K) - K^2}$$

EXAMPLE 1.6

Write the differential equations governing the mechanical rotational system shown in fig 1. and determine the transfer function $\theta(s)/T(s)$.

SOLUTION

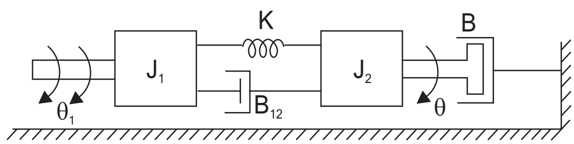


Fig 1.

In the given system, the torque T is the input and the angular displacement θ is the output.

Let, Laplace transform of $T = \mathcal{L}\{T\} = T(s)$

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\theta_1 = \mathcal{L}\{\theta_1\} = \theta_1(s)$

Hence the required transfer function is $\dfrac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The free body diagram of J_1 is shown in fig 2. The opposing torques acting on J_1 are marked as T_{j1}, T_{b12} and T_k.

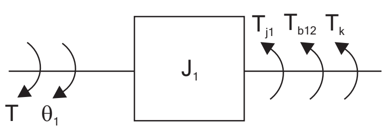


***Fig 2 :** Free body diagram of mass with moment of inertia J_1.*

$$T_{j1} = J_1\frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b12} = B_{12}\frac{d}{dt}(\theta_1 - \theta) \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $\quad T_{j1} + T_{b12} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12} \frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_1 s^2 \theta_1(s) + s\,B_{12}\,[\theta_1(s) - \theta(s)] + K\theta_1(s) - K\theta(s) = T(s)$$

$$\theta_1(s)\,[J_1 s^2 + s\,B_{12} + K\,] - \theta(s)\,[s\,B_{12} + K] = T(s) \qquad .....(1)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b12}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_{b12} = B_{12} \frac{d}{dt}(\theta - \theta_1)$$

$$T_b = B \frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

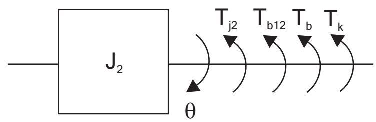


***Fig 3 :** Free body diagram of mass with moment of inertia J_2.*

By Newton's second law, $T_{j2} + T_{b12} + T_b + T_k = 0$

$$J_2 \frac{d^2\theta}{dt^2} + B_{12} \frac{d}{dt}(\theta - \theta_1) + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} - B_{12} \frac{d\theta_1}{dt} + \frac{d\theta}{dt}(B_{12} + B) + K\theta - K\theta_1 = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2 s^2 \theta(s) - B_{12}\, s\theta_1(s) + s\theta(s)\,[B_{12} + B] + K\theta(s) - K\theta_1(s) = 0$$

$$\theta(s)\,[s^2 J_2 + s(B_{12} + B) + K] - \theta_1(s)\,[sB_{12} + K] = 0$$

$$\theta_1(s) = \frac{[\,s^2 J_2 + s(B_{12} + B) + K\,]}{[sB_{12} + K]}\,\theta(s) \qquad .....(2)$$

Substituting for $\theta_1(s)$ from equation (2) in equation (1) we get,

$$[\,J_1 s^2 + sB_{12} + K\,]\,\frac{[J_2 s^2 + s(B_{12} + B) + K]\,\theta(s)}{(sB_{12} + K)} - (sB_{12} + K)\,\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + sB_{12} + K)\,[J_2 s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}{(sB_{12} + K)}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{(sB_{12} + K)}{(J_1 s^2 + sB_{12} + K)\,[J_2 s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}$$

RESULT

The differential equations governing the system are,

1. $J_1 \dfrac{d^2\theta_1}{dt^2} + B_{12} \dfrac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$

2. $J_2 \dfrac{d^2\theta_1}{dt^2} - B_{12} \dfrac{d\theta_1}{dt} + \dfrac{d\theta}{dt}(B_{12} + B) + K(\theta - \theta_1) = 0$

The transfer function of the system is,

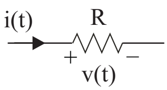
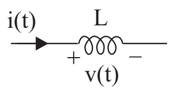
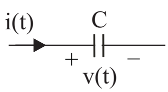
$$\frac{\theta(s)}{T(s)} = \frac{(sB_{12} + K)}{(J_1 s^2 + sB_{12} + K)\,[J_2 s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}$$

1.6 ELECTRICAL SYSTEMS

The models of electrical systems can be obtained by using resistor, capacitor and inductor. The current-voltage relation of resistor, inductor and capacitor are given in table-1. For modelling electrical systems, the electrical network or equivalent circuit is formed by using R, L and C and voltage or current source.

The differential equations governing the electrical systems can be formed by writing Kirchoff 's current law equations by choosing various nodes in the network or Kirchoff 's voltage law equations by choosing various closed paths in the network. The transfer function can be obtained by taking Laplace transform of the differential equations and rearranging them as a ratio of output to input.

Table-1.1 : Current-Voltage Relation of R, L and C

Element	Voltage across the element	Current through the element
	$v(t) = Ri(t)$	$i(t) = \dfrac{v(t)}{R}$
	$v(t) = L\,\dfrac{d}{dt}i(t)$	$i(t) = \dfrac{1}{L}\displaystyle\int v(t)\,dt$
	$v(t) = \dfrac{1}{C}\displaystyle\int i(t)\,dt$	$i(t) = C\dfrac{dv(t)}{dt}$

EXAMPLE 1.7

Obtain the transfer function of the electrical network shown in fig 1.

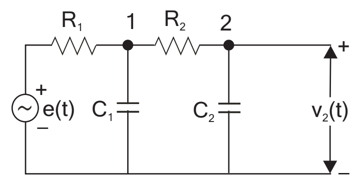


Fig 1.

SOLUTION

In the given network, input is e(t) and output is $v_2(t)$.

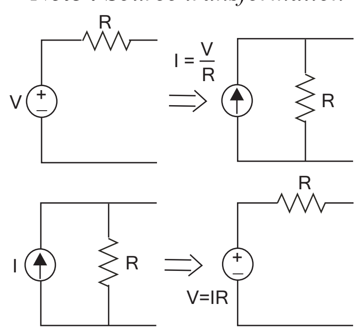
Let, Laplace transform of e(t) = $\mathcal{L}\{e(t)\}$ = E(s)

Laplace transform of $v_2(t)$ = $\mathcal{L}\{v_2(t)\}$ = $V_2(s)$

The transfer function of the network is $\dfrac{V_2(s)}{E(s)}$

Transform the voltage source in series with resistance R_1 into equivalent current source as shown in figure 2. The network has two nodes. Let the node voltages be v_1 and v_2. The Laplace transform of node voltages v_1 and v_2 are $V_1(s)$ and $V_2(s)$ respectively. The differential equations governing the network are given by the Kirchoff 's current law equations at these nodes.

Note : Source transformation



At node-1, by Kirchoff 's current law (refer fig 3)

$$\frac{v_1}{R_1} + C_1\frac{dv_1}{dt} + \frac{v_1 - v_2}{R_2} = \frac{e}{R_1}$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$\frac{V_1(s)}{R_1} + C_1 sV_1(s) + \frac{V_1(s)}{R_2} - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$V_1(s)\left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1} \qquad .....(1)$$

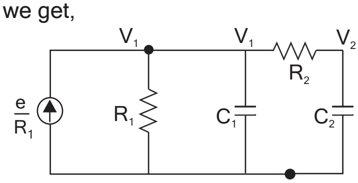


Fig 2.

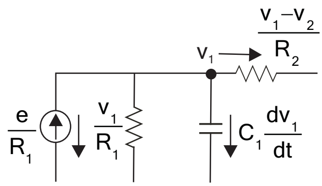


Fig 3.

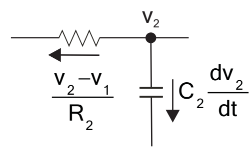


Fig 4.

At node-2, by Kirchoff's current law (refer fig 4)

$$\frac{v_2 - v_1}{R_2} + C_2\frac{dv_2}{dt} = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$\frac{V_2(s)}{R_2} - \frac{V_1(s)}{R_2} + C_2 s V_2(s) = 0$$

$$\frac{V_1(s)}{R_2} = \frac{V_2(s)}{R_2} + C_2 s V_2(s) = \left[\frac{1}{R_2} + sC_2\right]V_2(s)$$

$$\therefore\ V_1(s) = [1 + sC_2R_2]\,V_2(s) \qquad .....(2)$$

Substituting for $V_1(s)$ from equation (2) in equation (1) we get,

$$(1 + sR_2C_2)V_2(s)\left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$\left[\frac{(1 + sR_2C_2)(R_2 + R_1 + sC_1R_1R_2) - R_1}{R_1R_2}\right]V_2(s) = \frac{E(s)}{R_1}$$

$$\therefore\ \frac{V_2(s)}{E(s)} = \frac{R_2}{[(1 + sR_2C_2)(R_1 + R_2 + sC_1R_1R_2) - R_1]}$$

RESULT

The (node basis) differential equations governing the electrical network are,

1. $\frac{v_1}{R_1} + C_1\frac{dv_1}{dt} + \frac{v_1 - v_2}{R_2} = \frac{e}{R_1}$

2. $\frac{v_2 - v_1}{R_2} + C_2\frac{dv_2}{dt} = 0$

The transfer function of the electrical network is,

$$\frac{V_2(s)}{E(s)} = \frac{R_2}{[(1 + sR_2C_2)(R_1 + R_2 + sC_1R_1R_2) - R_1]}$$

1.7 TRANSFER FUNCTION OF ARMATURE CONTROLLED DC MOTOR

The speed of DC motor is directly proportional to armature voltage and inversely proportional to flux in field winding. In armature controlled DC motor the desired speed is obtained by varying the armature voltage. This speed control system is an electro-mechanical control system. The electrical system consists of the armature and the field circuit but for analysis purpose, only the armature circuit is considered because the field is excited by a constant voltage. The mechanical system consists of the rotating part of the motor and load connected to the shaft of the motor. The armature controlled DC motor speed control system is shown in fig 1.19.

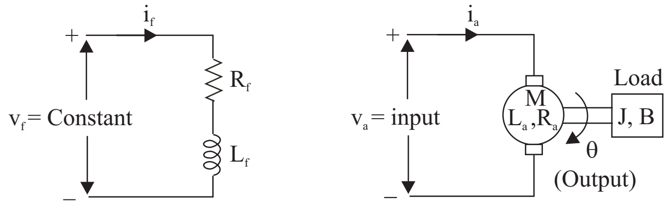


Fig 1.19 : *Armature controlled DC motor.*

Let, R_a = Armature resistance, Ω
L_a = Armature inductance, H
i_a = Armature current, A
v_a = Armature voltage, V
e_b = Back emf, V
K_t = Torque constant, N-m/A
T = Torque developed by motor, N-m
θ = Angular displacement of shaft, rad
ω = Angular velocity, rad/sec
J = Moment of inertia of motor and load, Kg-m^2/rad
B = Frictional coefficient of motor and load, N-m/(rad/sec)
K_b = Back emf constant, V/(rad/sec)

The equivalent circuit of armature is shown in fig 1.20.

By Kirchoff's voltage law, we can write,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \qquad .....(1.12)$$

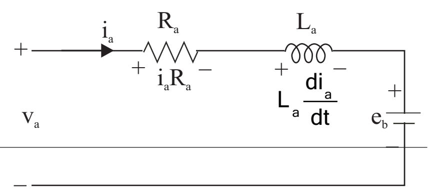


Fig 1.20 : *Equivalent circuit of armature.*

Torque of DC motor is proportional to the product of flux and current. Since flux is constant in this system, the torque is proportional to i_a alone.

$$T \propto i_a$$

$$\therefore \text{ Torque, } T = K_t\, i_a \qquad .....(1.13)$$

The mechanical system of the motor is shown in fig 1.21.

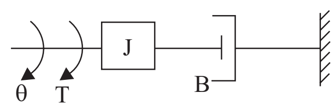


Fig 1.21.

The differential equation governing the mechanical system of motor is given by,

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \qquad .....(1.14)$$

The back emf of DC machine is proportional to speed (angular velocity) of shaft.

$$e_b \propto \omega \quad \text{and} \quad \omega = \frac{d\theta}{dt} \quad ; \quad \therefore\; e_b \propto \frac{d\theta}{dt} \quad \text{or} \quad \text{Back emf, } e_b = K_b\frac{d\theta}{dt} \qquad .....(1.15)$$

The Laplace transform of various time domain signals involved in this system are shown below.

$$\mathcal{L}\{v_a\} = V_a(s) \;;\; \mathcal{L}\{e_b\} = E_b(s) \;;\; \mathcal{L}\{T\} = T(s) \;;\; \mathcal{L}\{i_a\} = Ia(s) \;;\; \mathcal{L}\{\theta\} = \theta(s)$$

The differential equations governing the armature controlled DC motor speed control system are,

$$i_a R_a + L_a\frac{di_a}{dt} + e_b = v_a \;;\; T = K_t i_a \;;\; J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \;;\; e_b = K_b\frac{d\theta}{dt}$$

Taking Laplace transform of the above equations with zero initial conditions we get,

$$I_a(s)\,R_a + L_a s I_a(s) + E_b(s) = V_a(s) \qquad .....(1.16)$$

$$T(s) = K_t I_a(s) \qquad .....(1.17)$$

$$Js^2\theta(s) + B\,s\,\theta(s) = T(s) \qquad .....(1.18)$$

$$E_b(s) = K_b s\theta\,(s) \qquad .....(1.19)$$

On equating equations (1.17) and (1.18) we get,

$$K_t I_a(s) = (Js^2 + Bs)\,\theta(s)$$

$$I_a(s) = \frac{(Js^2 + Bs)}{K_t}\,\theta(s) \qquad .....(1.20)$$

Equation (1.16) can be written as,

$$(R_a + sL_a)\,I_a(s) + E_b(s) = V_a(s) \qquad .....(1.21)$$

Substituting for $E_b(s)$ and $I_a(s)$ from equation (1.19) and (1.20) respectively in equation (1.21),

$$(R_a + sL_a)\,\frac{(Js^2 + Bs)}{K_t}\,\theta(s) + K_b s\theta(s) = V_a(s)$$

$$\left[\frac{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s}{K_t}\right]\theta(s) = V_a(s)$$

The required transfer function is $\dfrac{\theta(s)}{V_a(s)}$

$$\therefore \frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s} \qquad ....(1.22)$$

$$= \frac{K_t}{R_a Js^2 + R_a Bs + L_a Js^3 + L_a Bs^2 + K_b K_t s}$$

$$= \frac{K_t}{s[\,JL_a s^2 + (JR_a + BL_a)s + (BR_a + K_b K_t)\,]}$$

$$= \frac{K_t/JL_a}{s\left[s^2 + \left(\dfrac{JR_a + BL_a}{JL_a}\right)s + \left(\dfrac{BR_a + K_b K_t}{JL_a}\right)\right]} \qquad .....(1.23)$$

The transfer function of armature controlled dc motor can be expressed in another standard form as shown below. From equation (1.22) we get,

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s} = \frac{K_t}{R_a\left(\dfrac{sL_a}{R_a} + 1\right)Bs\left(1 + \dfrac{Js^2}{Bs}\right) + K_b K_t s}$$

$$= \frac{K_t/R_a B}{s\left[(1 + sT_a)(1 + sT_m) + \dfrac{K_b K_t}{R_a B}\right]} \qquad .....(1.24)$$

where, $\frac{L_a}{R_a} = T_a$ = Electrical time constant

$\frac{J}{B} = T_m$ = Mechanical time constant

1.8 TRANSFER FUNCTION OF FIELD CONTROLLED DC MOTOR

The speed of a DC motor is directly proportional to armature voltage and inversely proportional to flux. In field controlled DC motor the armature voltage is kept constant and the speed is varied by varying the flux of the machine. Since flux is directly proportional to field current, the flux is varied by varying field current. The speed control system is an electromechanical control system. The electrical system consists of armature and field circuit but for analysis purpose, only field circuit is considered because the armature is excited by a constant voltage. The mechanical system consists of the rotating part of the motor and the load connected to the shaft of the motor. The field controlled DC motor speed control system is shown in fig 1.22.

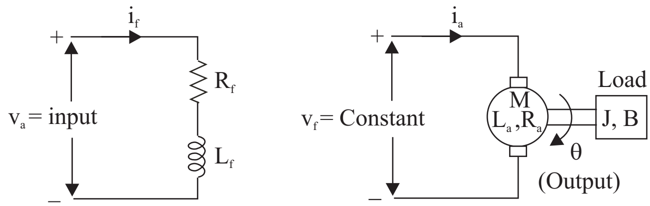


Fig 1.22 : *Field controlled DC motor.*

Let, R_f = Field resistance, Ω

L_f = Field inductance, H

i_f = Field current, A

v_f = Field voltage, V

T = Torque developed by motor, N-m

K_{tf} = Torque constant, N-m/A

J = Moment of inertia of rotor and load, Kg-m^2/rad

B = Frictional coefficient of rotor and load, N-m/(rad/sec)

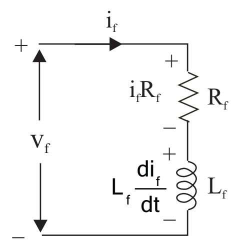


Fig 1.23 : *Equivalent circuit of field.*

The equivalent circuit of field is shown in fig 1.23.

By Kirchoff's voltage law, we can write

$$R_f i_f + L_f \frac{di_f}{dt} = v_f \quad .....(1.25)$$

The torque of DC motor is proportional to product of flux and armature current. Since armature current is constant in this system, the torque is proportional to flux alone, but flux is proportional to field current.

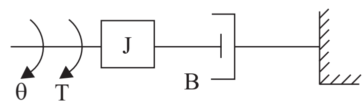


Fig 1.24.

$$T \propto i_f, \quad \therefore \text{Torque, } T = K_{tf}\, i_f \quad .....(1.26)$$

The mechanical system of the motor is shown in fig 1.24. The differential equation governing the mechanical system of the motor is given by,

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad .....(1.27)$$

The Laplace transform of various time domain signals involved in this system are shown below.

$\mathcal{L}\{i_f\} = I_f(s)$; $\mathcal{L}\{T\} = T(s)$; $\mathcal{L}\{v_f\} = V_f(s)$; $\mathcal{L}\{\theta\} = \theta(s)$

The differential equations governing the field controlled DC motor are,

$$R_f i_f + L_f \frac{di_f}{dt} = v_f \quad ; \quad T = K_{tf} i_f \quad ; \quad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T$$

On taking Laplace transform of the above equations with zero initial condition we get,

$$R_f I_f(s) + L_f s I_f(s) = V_f(s) \qquad .....(1.28)$$

$$T(s) = K_{tf} I_f(s) \qquad .....(1.29)$$

$$Js^2\theta(s) + B\,s\theta(s) = T(s) \qquad .....(1.30)$$

Equating equations (1.29) and (1.30) we get,

$$K_{tf} I_f(s) = Js^2\theta(s) + B\,s\theta(s)$$

$$I_f(s) = s\,\frac{(Js + B)}{K_{tf}}\,\theta(s) \qquad .....(1.31)$$

The equation (1.28) can be written as,

$$(R_f + sL_f)\,I_f(s) = V_f(s) \qquad .....(1.32)$$

On substituting for $I_f(s)$ from equation (1.31) in equation (1.32) we get,

$$(R_f + sL_f)\,s\,\frac{(Js + B)}{K_{tf}}\,\theta(s) = V_f(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{s(R_f + sL_f)(B + sJ)}$$

$$= \frac{K_{tf}}{sR_f\left(1 + \frac{sL_f}{R_f}\right)B\left(1 + \frac{sJ}{B}\right)} = \frac{K_m}{s(1 + sT_f)(1 + sT_m)} \qquad .....(1.33)$$

where, $K_m = \dfrac{K_{tf}}{R_f B}$ = Motor gain constant

$T_f = \dfrac{L_f}{R_f}$ = Field time constant

$T_m = \dfrac{J}{B}$ = Mechanical time constant

1.9 ELECTRICAL ANALOGOUS OF MECHANICAL TRANSLATIONAL SYSTEMS

Systems remain ***analogous*** as long as the differential equations governing the systems or transfer functions are in identical form. The electric analogue of any other kind of system is of greater importance since it is easier to construct electrical models and analyse them.

The three basic elements mass, dash-pot and spring that are used in modelling mechanical translational systems are analogous to resistance, inductance and capacitance of electrical systems.

The input force in mechanical system is analogous to either voltage source or current source in electrical systems. The output velocity (first derivative of displacement) in mechanical system is analogous to either current or voltage in an element in electrical system.

Since the electrical systems has two types of inputs either voltage or current source, there are two types of analogies : ***force-voltage analogy*** and ***force-current analogy***.

FORCE-VOLTAGE ANALOGY

The force balance equations of mechanical elements and their analogous electrical elements in force-voltage analogy are shown in table-1.2. The table-1.3 shows the list of analogous quantities in force-voltage analogy.

The following points serve as guidelines to obtain electrical analogous of mechanical systems based on force-voltage analogy.

1. In electrical systems the elements in series will have same current, likewise in mechanical systems, the elements having same velocity are said to be in series.
2. The elements having same velocity in mechanical system should have the same analogous current in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a closed loop in electrical system. A mass is considered as a node.
4. The number of meshes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of velocities of nodes (masses) in mechanical system.

Table- 1.2 : Analogous Elements in Force-Voltage Analogy

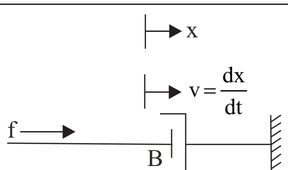
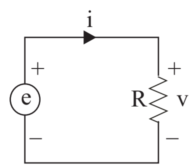
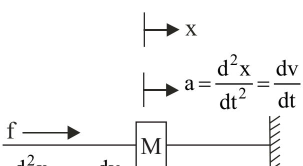
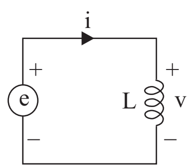
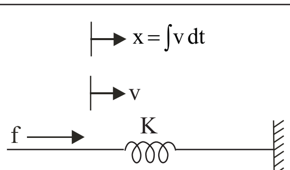
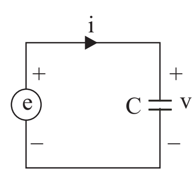
Mechanical system	**Electrical system**
Input : Force Output : Velocity	Input : Voltage source Output : Current through the element
 $f = B\frac{dx}{dt} = Bv$	 e = v and v=Ri $\therefore$ e = Ri
 $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $e = v$ and $v = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $f = Kx = K\int v\,dt$	 $e = v$ and $v = \frac{1}{C}\int i\,dt$ $\therefore e = \frac{1}{C}\int i\,dt$

Table -1.3 : Analogous Quantities in Force-Voltage Analogy

Item	Mechanical system	Electrical system (mesh basis system)
Independent variable (input)	Force, f	Voltage, e, v
Dependent variable (output)	Velocity, v	Current, i
	Displacement, x	Charge, q
Dissipative element	Frictional coefficient of dashpot, B	Resistance, R
Storage element	Mass, M	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\sum f = 0$	Kirchoff's voltage law $\sum v = 0$
Changing the level of independent variable	Lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

Table-1.4 : Analogous Elements in Force-Current Analogy

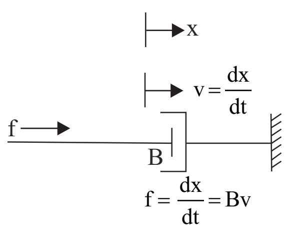
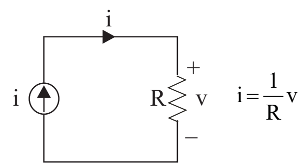
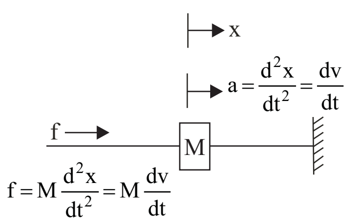
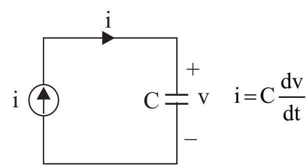
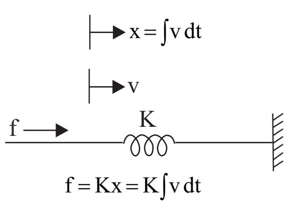
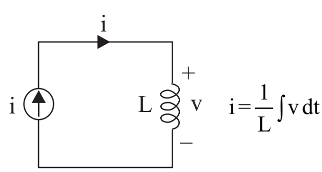
Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Current source Output : Voltage across the element
	
	
	

Table-1.5 : Analogous Quantities in Force-Current Analogy

Item	Mechanical system	Electrical system (node basis system)
Independent variable (input)	Force, f	Current, i
Dependent variable (output)	Velocity, v	Voltage, v
	Displacement, x	Flux, φ
Dissipative element	Frictional coefficient of dashpot, B	Conductance G=1/R
Storage element	Mass, M	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\sum f = 0$	Kirchoff's current law $\sum i = 0$
Changing the level of independent variable	Lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

5. The mechanical driving sources (force) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements in a closed loop in analogous electrical system.
6. The element connected between two (nodes) masses in mechanical system is represented as a common element between two meshes in electrical analogous system.

FORCE-CURRENT ANALOGY

The force balance equations of mechanical elements and their analogous electrical elements in force-current analogy are shown in table-1.4. The table-1.5 shows the list of analogous quantities in force-current analogy.

The following points serve as guidelines to obtain electrical analogous of mechanical systems based on force-current analogy.

1. In electrical systems elements in parallel will have same voltage, likewise in mechanical systems, the elements having same force are said to be in parallel.
2. The elements having same velocity in mechanical system should have the same analogous voltage in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. A mass is considered as a node.
4. The number of nodes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of velocities of (nodes) masses in mechanical system.
5. The mechanical driving sources (forces) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements connected to a node in electrical system.
6. The element connected between two nodes (masses) in mechanical system is represented as a common element between two nodes in electrical analogous system.

EXAMPLE 1.8

Write the differential equations governing the mechanical system shown in fig 1. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

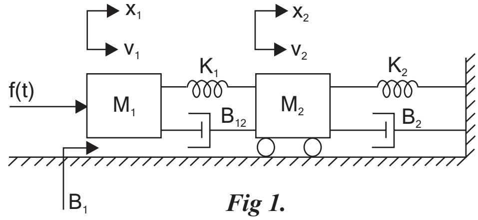


Fig 1.

SOLUTION

The given mechanical system has two nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1 and M_2 be x_1 and x_2 respectively. The corresponding velocities be v_1 and v_2.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2) \quad ; \quad f_{k1} = K_1(x_1 - x_2)$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$\therefore \; M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \qquad ......(1)$$

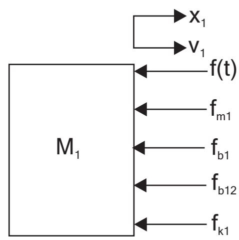


Fig 2.

The free body diagram of M_2 is shown in fig 3.The opposing forces are marked as f_{m2}, f_{b2}, f_{b12}, f_{k1} and f_{k2} .

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx_2}{dt} \quad ; \quad f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{k1} = K_1(x_2 - x_1) \quad ; \quad f_{k2} = K_2 x_2$$

By Newton's second law, $f_{m2} + f_{b2} + f_{k2} + f_{b12} + f_{k1} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_2 - x_1) = 0 \qquad .....(2)$$

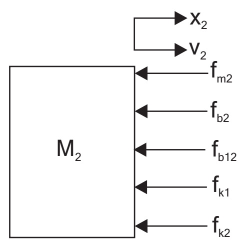


Fig 3.

On replacing the displacements by velocity in the differential equations (1) and (2) of the mechanical system we get,

$$\left(\text{i.e.,} \; \frac{d^2x}{dt^2} = \frac{dv}{dt} \quad ; \quad \frac{dx}{dt} = v \quad \text{and} \quad x = \int v \, dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + K_1 \int (v_1 - v_2)\, dt = f(t) \qquad .....(3)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + K_2 \int v_2 dt + B_{12}(v_2 - v_1) + K_1 \int (v_2 - v_1)\, dt = 0 \qquad .....(4)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-voltage analogous electrical circuit will have two meshes.

The force applied to mass, M_1 is represented by a voltage source in first mesh. The elements M_1, B_1, K_1 and B_{12} are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements K_1, B_{12}, M_2, K_2, and B_2 are connected to second node. Hence they are represented by analogous element in mesh-2 forming a closed path.

The elements K_1 and B_{12} are common between node-1 and 2 and so they are represented by analogous element as common elements between two meshes. The force-voltage electrical analogous circuit is shown in fig 4.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow e(t)$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$v_1 \rightarrow i_1$	$M_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$v_2 \rightarrow i_2$		$B_{12} \rightarrow R_{12}$	

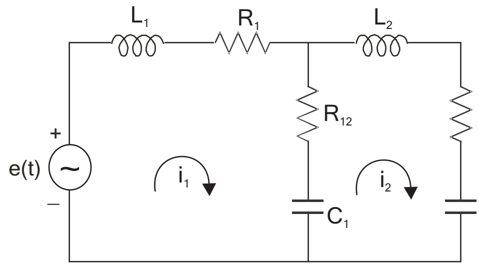


Fig 4 : *Force-voltage electrical analogous circuit.*

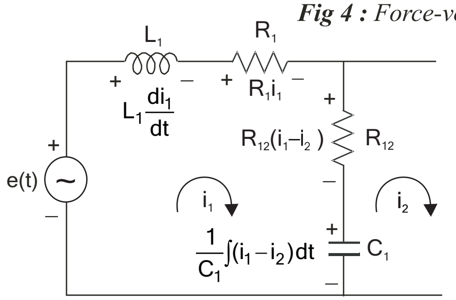


Fig 5 : *Mesh-1 of analogous circuit.*

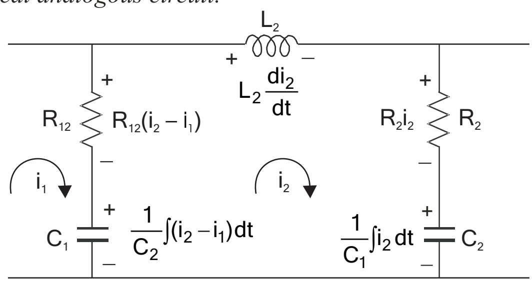


Fig 6 : *Mesh-2 of analogous circuit.*

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4 are given below (Refer fig 5 and 6).

$$L_1\frac{di_1}{dt} + R_1 i_1 + R_{12}(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\,dt = e(t) \qquad .....(5)$$

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1)\,dt = 0 \qquad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-current analogous electrical circuit will have two nodes.

The force applied to mass M_1 is represented as a current source connected to node-1 in analogous electrical circuit. The elements M_1, B_1, K_1 and B_{12} are connected to first node. Hence they are represented by analogous elements connected to node-1 in analogous electrical circuit. The elements K_1, B_{12}, M_2, K_2, and B_2 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit.

The elements K_1 and B_{12} are common between node-1 and 2 and so they are represented by analogous elements as common element between two nodes in analogous circuit. The force-current electrical analogous circuit is shown in fig 7.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$v_1 \rightarrow v_1$	$M_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_2 \rightarrow 1/L_2$
$v_2 \rightarrow v_2$	$B_{12} \rightarrow 1/R_{12}$		

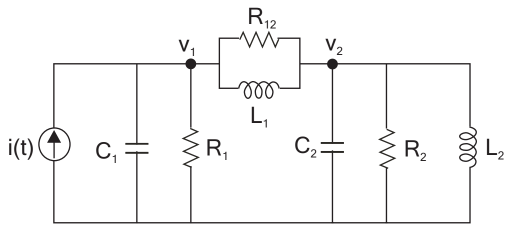


Fig 7 : *Force-voltage electrical analogous circuit.*

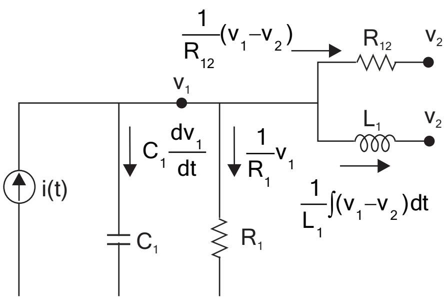


Fig 8 : *Node-1 of analogous circuit.*

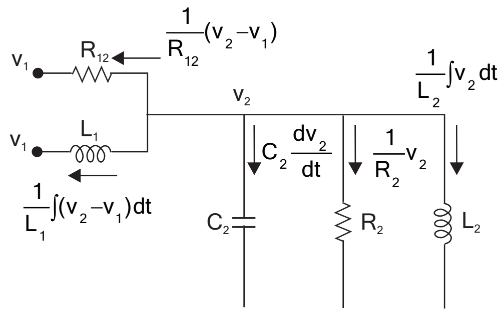


Fig 9 : *Node-2 of analogous circuit.*

The node basis equations using Kirchoff 's current law for the circuit shown in fig 7 are given below (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{R_{12}}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)dt=i(t) \quad .....(7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_{12}}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=0 \quad .....(8)$$

It is observed that the node basis equations (7) and (8) are similar to the differential equations (3) and (4) governing the mechanical system.

EXAMPLE 1.9

Write the differential equations governing the mechanical system shown in fig 1. Draw the force -voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

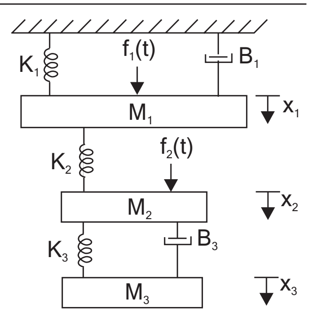


Fig 1.

SOLUTION

The given mechanical system has three nodes masses. The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1, M_2 and M_3 be x_1, x_2 and x_3 respectively. The corresponding velocities be v_1, v_2 and v_3.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{k2} and f_{k1}.

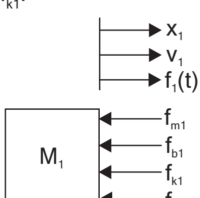


Fig 2.

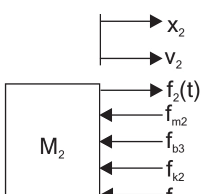


Fig 3.

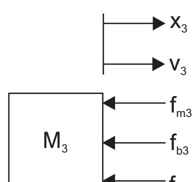


Fig 4.

$$f_{m1} = \frac{M_1 d^2 x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt} \quad ; \quad f_{k2} = K_2(x_1 - x_2) \quad ; \quad f_{k1} = K_1 x_1$$

By Newton's second law, $f_{m1} + f_{b1} + f_{k2} + f_{k1} = f_1(t)$

$$M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_2(x_1 - x_2) + K_1 x_1 = f_1(t) \qquad .....(1)$$

Free body diagram of M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b3}, f_{k2} f_{k3}.

$$f_{m2} = M_2 \frac{d^2 x_2}{dt^2} \quad ; \quad f_{b3} = B_3 \frac{d}{dt}(x_2 - x_3) \quad ; \quad f_{k2} = K_2(x_2 - x_1) \quad ; \quad f_{k3} = K_3(x_2 - x_3)$$

By Newton's second law,

$$M_2 \frac{d^2 x_2}{dt^2} + B_3 \frac{d}{dt}(x_2 - x_3) + K_2(x_2 - x_1) + K_3(x_2 - x_3) = f_2(t) \qquad .....(2)$$

The free body diagram of M_3 is shown in fig 4. The opposing forces are marked as f_{m3}, f_{b3} and f_{k3}.

$$f_{m3} = M_3 \frac{d^2 x_3}{dt^2} \quad ; \quad f_{b3} = B_3 \frac{d}{dt}(x_3 - x_2) \quad ; \quad f_{k3} = K_3(x_3 - x_2)$$

By Newton's second law,

$$M_3 \frac{d^2 x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_2) + K_3(x_3 - x_2) = 0 \qquad .....(3)$$

On replacing the displacements by velocity in the differential equations (1), (2) and (3) governing the mechanical system we get,

$$\left(\text{i.e.,} \frac{d^2 x}{dt^2} = \frac{dv}{dt} \quad ; \quad \frac{dx}{dt} = v \ \text{ and } \ x = \int v \, dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + K_1 \int v_1 dt + K_2 \int (v_1 - v_2)\, dt = f_1(t) \qquad .....(4)$$

$$M_2 \frac{dv_2}{dt} + B_3(v_2 - v_3) + K_2 \int (v_2 - v_1)\, dt + K_3 \int (v_2 - v_3)\, dt = f_2(t) \qquad .....(5)$$

$$M_3 \frac{dv_3}{dt} + B_3(v_3 - v_2) + K_3 \int (v_3 - v_2)\, dt = 0 \qquad .....(6)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-voltage analogous electrical circuit will have three meshes. The force applied to mass, M_1 is represented by a voltage source in first mesh and the force applied to mass, M_2 is represented by a voltage source in second mesh.

The elements M_1, B_1, K_1 and K_2 are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements M_2, B_3, K_2 and K_3 are connected to second node. Hence they are represented by analogous element in mesh-2 forming a closed path. The elements M_3, K_3 and B_3 are connected to third node. Hence they are represented by analogous element in mesh-3 forming a closed path.

The element K_2 is common between node-1 and 2 and so it is represented by analogous element as common element between mesh 1 and 2. The elements K_3 and B_3 are common between node-2 and 3 and so they are represented by analogous elements as common elements between mesh-2 and 3. The force-voltage electrical analogous circuit is shown in fig 5.

The electrical analogous elements for the elements of mechanical system are given below.

$f_1(t) \rightarrow e_1(t)$	$v_1 \rightarrow i_1$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$f_2(t) \rightarrow e_2(t)$	$v_2 \rightarrow i_2$	$M_2 \rightarrow L_2$	$B_3 \rightarrow R_3$	$K_2 \rightarrow 1/C_2$
	$v_3 \rightarrow i_3$	$M_3 \rightarrow L_3$		$K_3 \rightarrow 1/C_3$

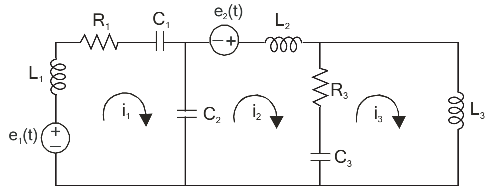


Fig 5 : Force-voltage electrical analogous circuit.

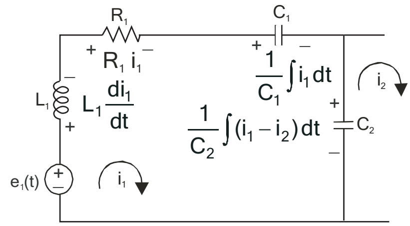


Fig 6 : Mesh-1 analogous circuit.

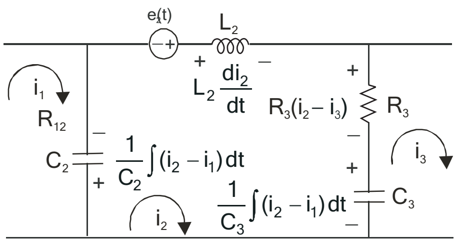


Fig 7 : Mesh-2 of analogous circuit.

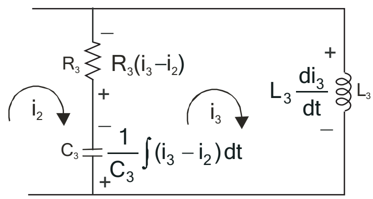


Fig 8 : Mesh-3 of analogous circuit.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 5 are given below (Refer fig 6, 7, 8).

$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int i_1dt+\frac{1}{C_2}\int(i_1-i_2)\,dt=e_1(t) \qquad .....(7)$$

$$L_2\frac{di_2}{dt}+R_3(i_2-i_3)+\frac{1}{C_3}\int(i_2-i_3)\,dt+\frac{1}{C_2}\int(i_2-i_1)\,dt=e_2(t) \qquad .....(8)$$

$$L_3\frac{di_3}{dt}+R_3(i_3-i_2)+\frac{1}{C_3}\int(i_3-i_2)\,dt=0 \qquad .....(9)$$

It is observed that the mesh equations (7), (8) and (9) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will have three nodes.

The force applied to mass M_1 is represented as a current source connected to node-1 in analogous electrical circuit. The force applied to mass M_2 is represented as a current source connected to node-2 in analogous electrical circuit.

The elements M_1, B_1, K_1 and K_2 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements M_2, B_3, K_2 and K_3 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit. The elements M_3, B_3 and K_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node-3 in analogous electrical circuit.

The element K_2 is common between node-1 and 2 and so it is represented by analogous element as common element between node-1 and 2 in analogous circuit. The elements B_3 and K_3 are common between node-2 and 3 and so they are represented by analogous elements as common elements between node-2 and 3. The force-current electrical analogous circuit is shown in fig 9.

The electrical analogous elements for the elements of mechanical system are given below.

$f_1(t) \to i_1(t)$	$v_1 \to v_1$	$M_1 \to C_1$	$B_1 \to 1/R_1$	$K_1 \to 1/L_1$
$f_2(t) \to i_2(t)$	$v_2 \to v_2$	$M_2 \to C_2$	$B_3 \to 1/R_3$	$K_2 \to 1/L_2$
	$v_3 \to v_3$	$M_3 \to C_3$		$K_3 \to 1/L_3$

The node basis equations using Kirchoff 's current law for the circuit shown in fig 9. are given below. (Refer fig 10, 11,12).

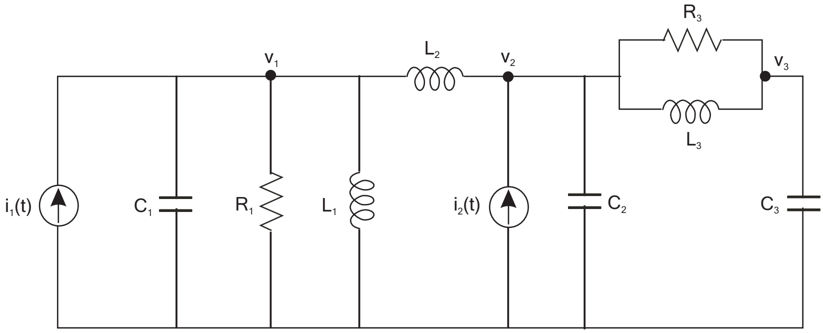


Fig 9 : *Force-current electrical analogous circuit.*

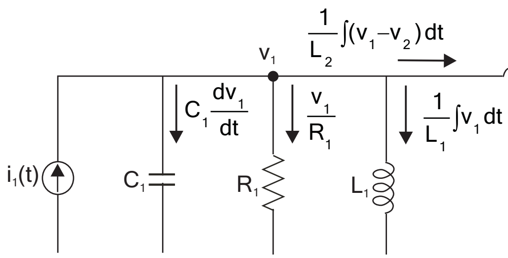


Fig 10 : *Node-1 of analogous circuit.*

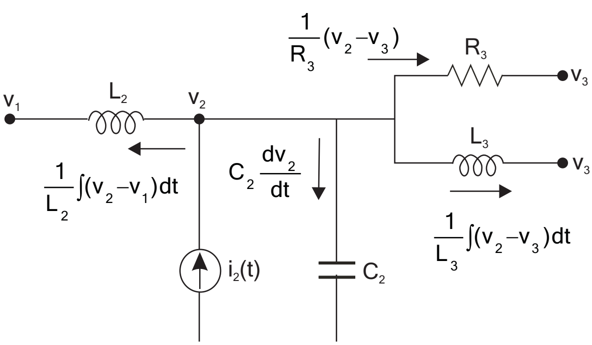


Fig 11 : *Node-2 of analogous circuit.*

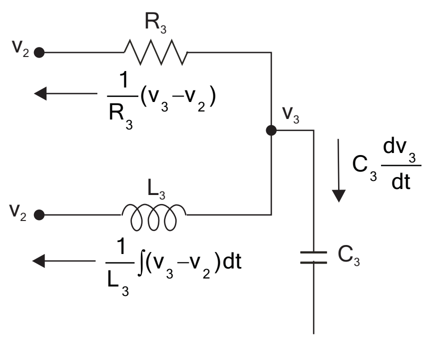


Fig 12 : *Node-3 of analogous circuit.*

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int v_1 dt+\frac{1}{L_2}\int(v_1-v_2)\,dt = i_1(t) \qquad .....(10)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_3}(v_2-v_3)+\frac{1}{L_3}\int(v_2-v_3)\,dt+\frac{1}{L_2}\int(v_2-v_1)\,dt = i_2(t) \qquad .....(11)$$

$$C_3\frac{dv_3}{dt}+\frac{1}{R_3}(v_3-v_2)+\frac{1}{L_3}\int(v_3-v_2)\,dt = 0 \qquad .....(12)$$

It is observed that node basis equations (10), (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

EXAMPLE 1.10

Write the differential equations governing the mechanical system shown in fig 1.Draw force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

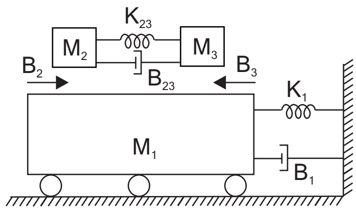


Fig 1.

SOLUTION

The given mechanical system has three nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1, M_2 and M_3 be x_1, x_2 and x_3 respectively. The corresponding velocities be v_1,v_2 and v_3.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{b1}, f_{k1}, f_{b2}, f_{b3}, and f_{m1}.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt} \quad ; \quad f_{k1} = K_1 x_1$$

$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x_2) \quad ; \quad f_{b3} = B_3 \frac{d}{dt}(x_1 - x_3)$$

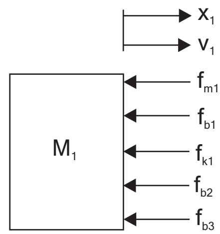


Fig 2.

By Newton's second law, $f_{m1} + f_{b1} + f_{k1} + f_{b2} + f_{b3} = 0$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 x_1 + B_2 \frac{d}{dt}(x_1 - x_2) + B_3 \frac{d}{dt}(x_1 - x_3) = 0 \quad .....(1)$$

The free body diagram of M_2 is shown in fig 3. The opposing forces are marked as f_{m2} , f_{b2} , f_{b23} and f_{k23}.

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{d}{dt}(x_2 - x_1)$$

$$f_{b23} = B_{23} \frac{d}{dt}(x_2 - x_3) \quad ; \quad f_{k23} = (x_2 - x_3)$$

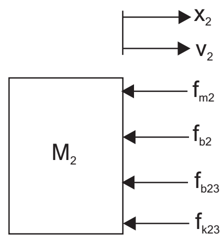


Fig 3.

By Newton's second law, $f_{m2} + f_{b2} + f_{b23} + f_{k23} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{d}{dt}(x_2 - x_1) + B_{23} \frac{d}{dt}(x_2 - x_3) + K_{23}(x_2 - x_3) = 0 \quad ......(2)$$

The free body diagram of M_3 is shown in fig 4. The opposing forces are marked as f_{m3} , f_{b3} , f_{b23} , and f_{k23}.

$$f_{m3} = M_3 \frac{d^2x_3}{dt^2} \quad ; \quad f_{b3} = B_3 \frac{d}{dt}(x_3 - x_1)$$

$$f_{b23} = B_{23} \frac{d}{dt}(x_3 - x_2) \quad ; \quad f_{k23} = K_{23}(x_3 - x_2)$$

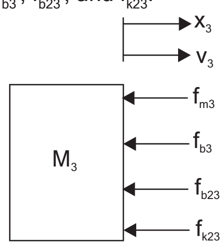


Fig 4.

By Newton's second law, $f_{m3} + f_{b3} + f_{b23} + f_{k23} = 0$

$$M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_1) + B_{23} \frac{d}{dt}(x_3 - x_2) + K_{23}(x_3 - x_2) = 0 \quad .....(3)$$

On replacing the displacements by velocity in the differential equations (1), (2) and (3) governing the mechanical system we get,

$$\left(\text{i.e.,} \ \frac{d^2x}{dt^2} = \frac{dv}{dt}, \ \frac{dx}{dt} = v \ \text{ and } \ x = \int v dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + K_1 \int v_1 dt + B_2(v_1 - v_2) + B_3(v_1 - v_3) = 0 \quad .....(4)$$

$$M_2 \frac{dv_2}{dt} + B_2(v_2 - v_1) + B_{23}(v_2 - v_3) + K_{23} \int (v_2 - v_3) dt = 0 \quad .....(5)$$

$$M_3 \frac{dv_3}{dt} + B_3(v_3 - v_1) + B_{23}(v_3 - v_2) + K_{23} \int (v_3 - v_2) dt = 0 \quad .....(6)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-voltage analogous electrical circuit will have three meshes. Since there is no applied force in mechanical system there will not be any voltage source in analogous electrical circuit.

The elements M_1, K_1, B_1, B_3 and B_2 are connected to first node. Hence they are represented by analogous elements in mesh-1 forming a closed path. The elements M_2, K_{23}, B_{23} and B_2 are connected to second node. Hence they are represented by analogous elements in mesh-2 forming a closed path. The elements M_3, K_{23} , B_{23} and B_3 are connected to third node. Hence they are represented by analogous elements in mesh-3 forming a closed path.

The elements K_{23} and B_{23} are common between node-2 and 3 and so they are represented by analogous element as common elements between mesh-2 and 3. The element B_2 is common between node-1 and 2 and so it is represented by analogous element as common element between mesh-1 and 2. The element B_3 is common between node-1 and 3 and so it is represented by analogous element between mesh-1 and 3. The force-voltage electrical analogous circuit is shown in fig 5.

The electrical analogous elements for the elements of mechanical system are given below.

$v_1 \to i_1$	$M_1 \to L_1$	$K_1 \to 1/C_1$	$B_2 \to R_2$
$v_2 \to i_2$	$M_2 \to L_2$	$K_{23} \to 1/C_{23}$	$B_3 \to R_3$
$v_3 \to i_3$	$M_3 \to L_3$	$B_1 \to R_1$	$B_{23} \to R_{23}$

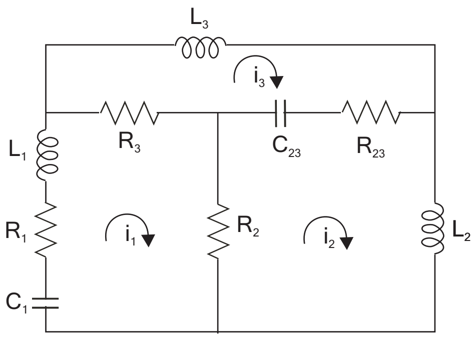


Fig 5 : *Force-voltage electrical analogous circuit.*

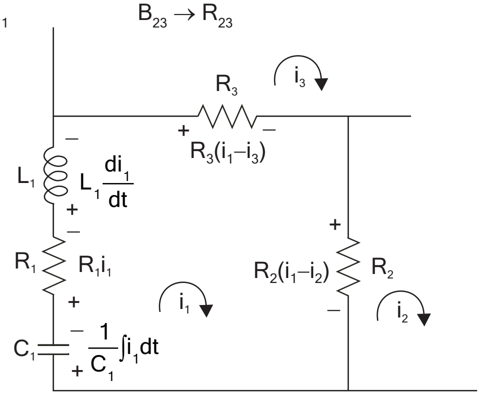


Fig 6 : *Mesh-1 of analogous circuit.*

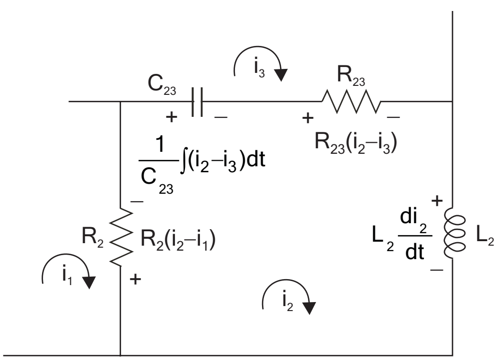


Fig 7 : *Mesh-2 of analogous circuit.*

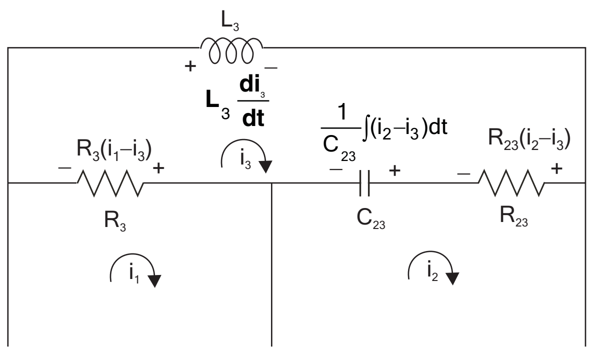


Fig 8 : *Mesh-3 of analogous circuit..*

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 5 are given below. (Refer fig 6,7 and 8).

$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int i_1dt+R_2(i_1-i_2)+R_3(i_1-i_3)=0 \qquad .....(7)$$

$$L_2\frac{di_2}{dt}+R_2(i_2-i_1)+\frac{1}{C_{23}}\int (i_2-i_3)\,dt+R_{23}(i_2-i_3)=0 \qquad .....(8)$$

$$L_3\frac{di_3}{dt}+R_3(i_3-i_1)+\frac{1}{C_{23}}\int(i_3-i_2)\,dt+R_{23}(i_3-i_2)=0 \qquad .....(9)$$

It is observed that the mesh basis equations (7), (8) and (9) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will have three nodes. Since there is no applied force in mechanical system there will not be any current source in analogous electrical circuit.

The elements M_1, K_1, B_1, B_2 and B_3 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements M_2, K_{23}, B_{23} and B_2 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit. The elements M_3, K_{23} , B_{23} and B_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node-3 in analogous electrical circuit.

The elements K_{23} and B_{23} are common between node-2 and 3 and so they are represented by analogous element as common elements between node-2 and 3 in electrical analogous circuit. The element B_2 is common between node-1 and 2 and so it is represented by analogous element as common element between node-1 and 2 in electrical analogous circuit. The element B_3 is common between node-1 and 3 and so it is represented by analogous element as common element between node-1 and 3 in electrical analogous circuit. The force-current electrical analogous circuit is shown in fig 9.

The electrical analogous elements for the elements of mechanical system are given below.

$v_1 \rightarrow v_1$	$M_1 \rightarrow C_1$	$K_1 \rightarrow 1/L_1$	$B_2 \rightarrow 1/R_2$
$v_2 \rightarrow v_2$	$M_2 \rightarrow C_2$	$K_{23} \rightarrow 1/L_{23}$	$B_3 \rightarrow 1/R_3$
$v_3 \rightarrow v_3$	$M_3 \rightarrow C_3$	$B_1 \rightarrow 1/R_1$	$B_{23} \rightarrow 1/R_{23}$

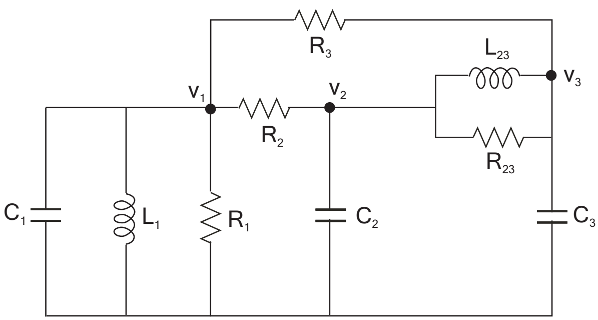


Fig 9 : *Force-current electrical analogous circuit.*

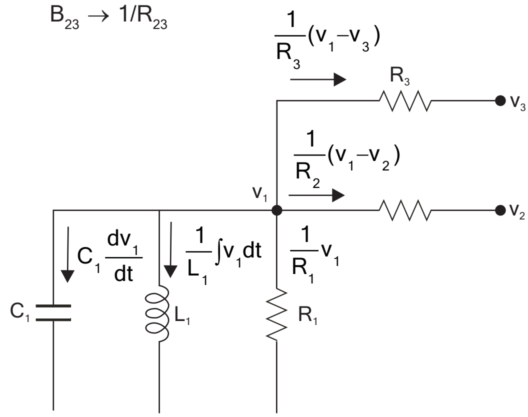


Fig 10 : *Node-1 of analogous circuit.*

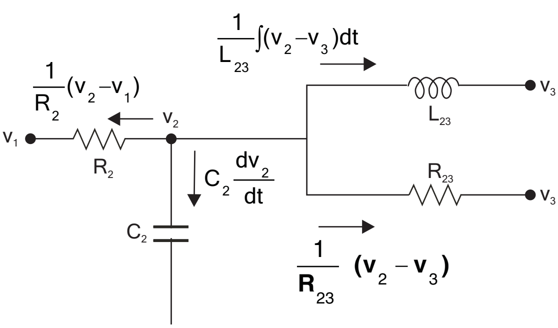


Fig 11 : *Node-2 of analogous circuit.*

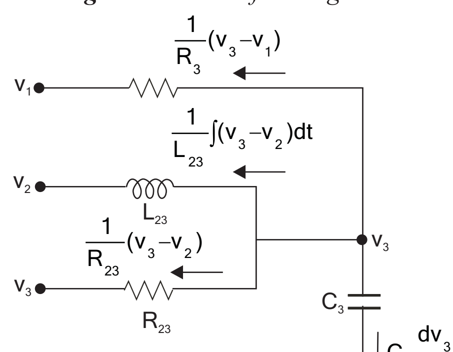


Fig 12 : *Node-3 of analogous circuit.*

The node basis equations using Kirchoff's current law for the circuit shown in fig 9 are given below. (Refer fig 10, 11 and 12).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int v_1dt+\frac{1}{R_2}(v_1-v_3)+\frac{1}{R_3}(v_1-v_3)=0 \quad .....(10)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}(v_2-v_1)+\frac{1}{L_{23}}\int(v_2-v_3)dt+\frac{1}{R_{23}}(v_2-v_3)=0 \quad .....(11)$$

$$C_3\frac{dv_3}{dt}+\frac{1}{R_3}(v_3-v_1)+\frac{1}{L_{23}}\int(v_3-v_2)dt+\frac{1}{R_{23}}(v_3-v_2)=0 \quad .....(12)$$

It is observed that the node basis equations (10), (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

EXAMPLE 1.11

Write the differential equations governing the mechanical system shown in fig 1. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

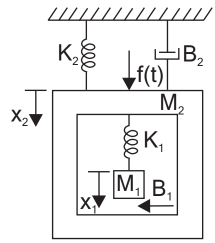


Fig 1.

SOLUTION

The given mechanical system has two nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacement of masses M_1 and M_2 be x_1 and x_2 respectively. The corresponding velocities be v_1 and v_2.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1} and f_{k1}.

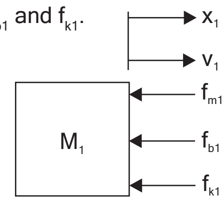


Fig 2.

$$f_{m1}=M_1\frac{d^2x_1}{dt^2} \quad ; \quad f_{b1}=B_1\frac{d(x_1-x_2)}{dt} \quad ; \quad f_{k1}=K_1(x_1-x_2)$$

By Newton's second law, $f_{m1}+f_{b1}+f_{k1}=0$

$$M_1\frac{d^2x_1}{dt^2}+B_1\frac{d(x_1-x_2)}{dt}+K_1(x_1-x_2)=0 \quad .....(1)$$

The free body diagram of M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b1}, f_{k2} and f_{k1}.

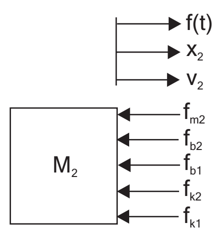


Fig 3.

$$f_{m2}=M_2\frac{d^2x_2}{dt^2} \quad ; \quad f_{b2}=B_2\frac{dx_2}{dt} \quad ; \quad f_{b1}=B_1\frac{d}{dt}(x_2-x_1)$$

$$f_{k2}=K_2x_2 \quad ; \quad f_{k1}=K_1(x_2-x_1)$$

By Newton's second law, $f_{m2}+f_{b2}+f_{k2}+f_{b1}+f_{k1}=f(t)$

$$M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+K_2x_2+B_1\frac{d}{dt}(x_2-x_1)+K_1(x_2-x_1)=f(t) \quad .....(2)$$

On replacing the displacements by velocity in the differential equations (1) and (2) governing the mechanical system we get,

$$\left(\text{i.e.,}\ \frac{d^2x}{dt^2}=\frac{dv}{dt}\ ,\ \frac{dx}{dt}=v\ \text{and}\ x=\int v\,dt\right)$$

$$M_1\frac{dv_1}{dt}+B_1(v_1-v_2)+K_1\int(v_1-v_2)\,dt=0 \quad .....(3)$$

$$M_2\frac{dv_2}{dt}+B_2v_2+K_2\int v_2dt+B_1(v_2-v_1)+K_1\int(v_2-v_1)dt=f(t) \quad .....(4)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force voltage analogous electrical circuit will have two meshes. The force applied to mass, M_2 is represented by a voltage source in second mesh.

The elements M_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous element in mesh 1 forming a closed path. The elements M_2, K_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous element in mesh 2 forming a closed path.

The elements B_1 and K_1 are common between node 1 and 2 and so they are represented as common elements between mesh 1 and 2. The force-voltage electrical analogous circuit is shown in fig 4.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow e(t)$ $\quad v_1 \rightarrow i_1 \quad M_1 \rightarrow L_1 \quad K_1 \rightarrow 1/C_1 \quad B_1 \rightarrow R_1$

$\quad v_2 \rightarrow i_2 \quad M_2 \rightarrow L_2 \quad K_2 \rightarrow 1/C_2 \quad B_2 \rightarrow R_2$

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4. are given below, (refer fig 5 and 6).

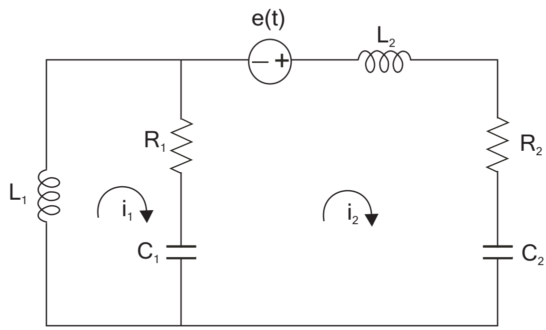


Fig 4 : *Force-voltage electrical analogous circuit.*

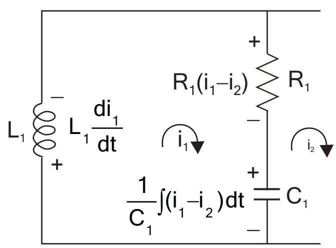


Fig 5 : *Mesh-1 of analogous circuit.*

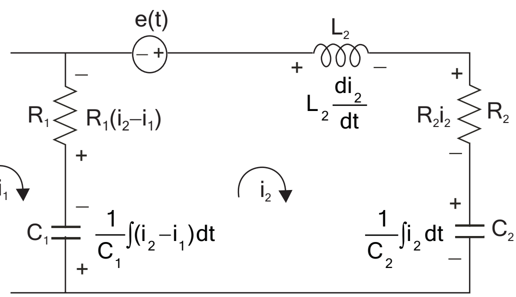


Fig 6 : *Mesh-2 of analogous circuit.*

$$L_1\frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\,dt = 0 \qquad .....(5)$$

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2 dt + \frac{1}{C_1}\int (i_2 - i_1)\,dt + R_1(i_2 - i_1) = e(t) \qquad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-current analogous electrical circuit will have two nodes. The force applied to mass M_2 is represented as a current source connected to node-2 in analogous electrical circuit.

The elements M_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements M_2, K_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit.

The elements K_1 and B_1 is common to node-1 and 2 and so they are represented by analogous element as common elements between two nodes in analogous circuit. The force-current electrical analogous circuit is shown in fig 7.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$	$v_1 \rightarrow v_1$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
	$v_2 \rightarrow v_2$	$M_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_2 \rightarrow 1/L_2$

The node basis equations using Kirchoff 's current law for the circuit shown in fig.7, are given below, (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}(v_1 - v_2) + \frac{1}{L_1}\int (v_1 - v_2)\,dt = 0 \qquad .....(7)$$

$$C_2\frac{dv_2}{dt} + \frac{1}{R_2}v_2 + \frac{1}{L_2}\int v_2 dt + \frac{1}{R_1}(v_2 - v_1) + \frac{1}{L_1}\int (v_2 - v_1) = i(t) \qquad .....(8)$$

It is observed that the node basis equations (7) and (8) are similar to the differential equations (3) and (4) governing the mechanical system.

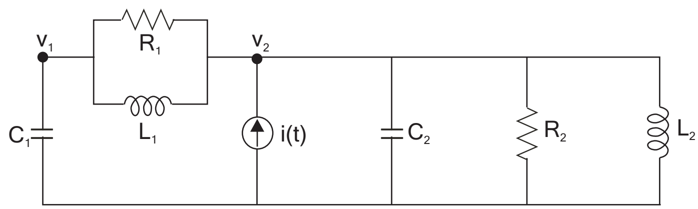


***Fig 7 :** Force-current electrical analogous circuit.*

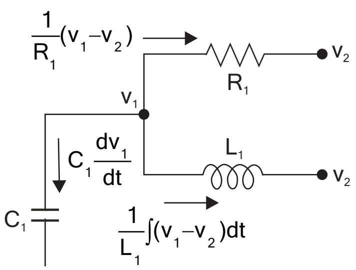


***Fig 8 :** Node-1 of analogous circuit.*

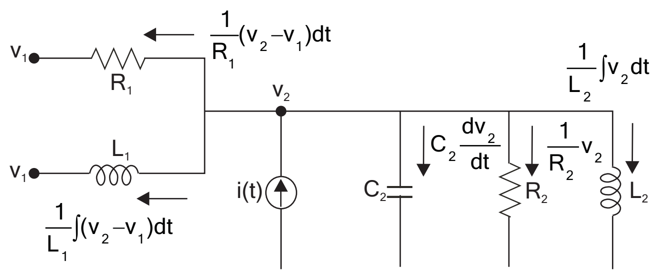


***Fig 9 :** Node-2 of analogous circuit.*

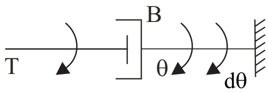
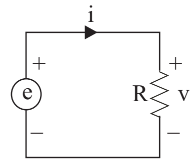
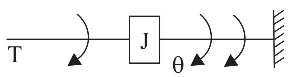
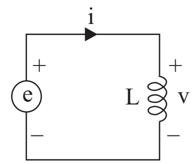
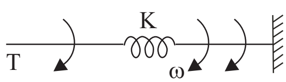
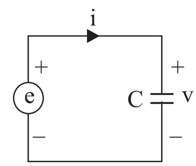
1.10 ELECTRICAL ANALOGOUS OF MECHANICAL ROTATIONAL SYSTEMS

The three basic elements moment of inertia, rotational dashpot and torsional spring that are used in modelling mechanical rotational systems are analogous to resistance, inductance and capacitance of electrical systems. The input torque in mechanical system is analogous to either voltage source or current source in electrical systems. The output angular velocity (first derivative of angular displacement) in mechanical rotational system is analogous to either current or voltage in an element in electrical system. Since the electrical systems has two types of inputs either voltage source or current source, there are two types of analogies: ***torque-voltage analogy and torque-current analogy***.

TORQUE-VOLTAGE ANALOGY

The torque balance equations of mechanical rotational elements and their analogous electrical elements in torque-voltage analogy are shown in table-1.6. The table-1.7 shows the list of analogous quantities in torque-voltage analogy.

TABLE-1.6 : Analogous Element of Torque-Voltage Analogy

Mechanical rotational system	Electrical system
Input : Torque Output : Angular velocity	Input : Voltage source Output : Current through the element
 $\omega = \frac{d\theta}{dt}$ $T = B\frac{d\theta}{dt} = B\omega$	 $e = v \ ; \ v = Ri$ $\therefore \ e = Ri$
 $T = J\frac{d^2\theta}{dt^2} = j\frac{d\omega}{dt}$ $\quad \propto = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$	 $e = v \ ; \ v = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $\theta = \int \omega \, dt$ $T = K\theta = K\int \omega \, dt$	 $e = v \ ; \ v = \frac{1}{C}\int i \, dt$ $\therefore e = \frac{1}{C}\int i \, dt$

The following points serve as guidelines to obtain electrical analogous of mechanical rotational systems based on torque-voltage analogy.

1. In electrical systems the elements in series will have same current, likewise in mechanical systems, the elements having same angular velocity are said to be in series.
2. The elements having same angular velocity in mechanical system should have analogous same current in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a closed loop in electrical system. The moment of inertia of mass is considered as a node.
4. The number of meshes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of angular velocities of nodes (moment of inertia of mass) in mechanical system.
5. The mechanical driving sources (Torque) and passive elements connected to the node (moment of inertia of mass) in mechanical system should be represented by analogous element in a closed loop in analogous electrical system.
6. The element connected between two nodes (moment of inertia) in mechanical system is represented as a common element between two meshes in electrical analogous system.

Table-1.7 : Analogous Quantities in Torque-Voltage Analogy

Item	Mechanical rotational system	Electrical system (mesh basis system)
Independent variable (input)	Torque, T	Voltage, e, v
Dependent variable (output)	Angular Velocity, ω	Current, i
	Angular displacement, θ	Charge, q
Dissipative element	Rotational coefficient of dashpot, B	Resistance, R
Storage element	Moment of inertia, J	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\sum T = 0$	Kirchoff's voltage law $\sum v = 0$
Changing the level of independent variable	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

TORQUE-CURRENT ANALOGY

The torque balance equations of mechanical elements and their analogous electrical elements in torque-current analogy are shown in table-1.8. The table-1.9 shows the list of analogous quantities in torque-current analogy.

The following points serve as guidelines to obtain electrical analogous of mechanical rotational systems based on Torque-current analogy.

1. In electrical systems the elements in parallel will have same voltage, likewise in mechanical systems, the elements having same torque are said to be in parallel.
2. The elements having same angular velocity in mechanical system should have analogous same voltage in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. The moment of inertia of mass is considered as a node.
4. The number of nodes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of angular velocities of nodes (moment of inertia of mass) in mechanical system.
5. The mechanical driving sources (Torque) and passive elements connected to the node in mechanical system should be represented by analogous element connected to a node in analogous electrical system.
6. The element connected between two nodes (moment of inertia of mass) in mechanical system is represented as a common element between two nodes in electrical analogous system.

TABLE-1.8 : Analogous Elements in Torque-Current Analogy

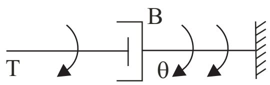
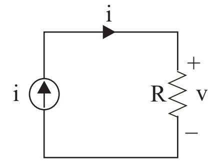
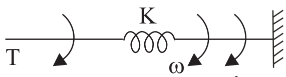
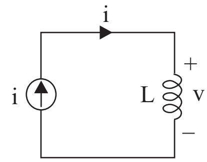
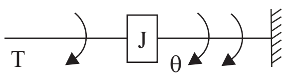
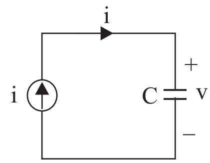
Mechanical rotational system	Electrical system
Input : Torque Output : Angular velocity	Input : Current source Output : Voltage across the element
 $T = B\frac{d\theta}{dt} = B\omega$ $\quad \omega = \frac{d\theta}{dt}$	 $i = \frac{1}{R}v$
 $T = K\theta = K\int \omega\, dt$ $\quad \theta = \int \omega\, dt$	 $i = \frac{1}{L}\int v\, dt$
 $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$ $\quad \propto = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$	 $i = C\frac{dv}{dt}$

Table-1.9 : Analogous Quantities in Torque-Current Analogy

Item	Mechanical rotational system	Electrical system (node basis system)
Independent variable (input)	Torque, T	Current, i
Dependent variable (output)	Angular Velocity, ω	Voltage, v
	Angular displacement, θ	Flux, ϕ
Dissipative element	Rotational frictional coefficient of dashpot, B	Conductance, G = 1/R
Storage element	Moment of inertia, J	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\sum T = 0$	Kirchoff 's current law $\sum i = 0$
Changing the level of independent variable	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

EXAMPLE 1.12

Write the differential equations governing the mechanical rotational system shown in fig 1. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equations.

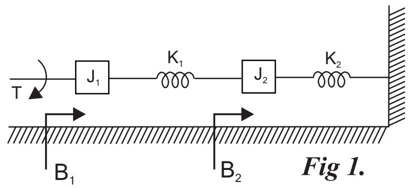


Fig 1.

SOLUTION

The given mechanical rotational system has two nodes (moment of inertia of masses). The differential equations governing the mechanical rotational system are given by torque balance equations at these nodes.

Let the angular displacements of J_1 and J_2 be θ_1 and θ_2 respectively. The corresponding angular velocities be ω_1 and ω_2.

The free body diagram of J_1 is shown in fig 2. The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b1} = B_1 \frac{d\theta_1}{dt} \quad ; \quad T_{k1} = K_1(\theta_1 - \theta_2)$$

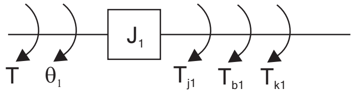


Fig 2.

By Newton's second law, $T_{j1} + T_{b1} + T_{k1} = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) = T \qquad .....(1)$$

The free body diagram of J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b2}, T_{k2} and T_{k1}.

$$T_{j2} = J_2 \frac{d^2\theta_2}{dt^2} \quad ; \quad T_{b2} = B_2 \frac{d\theta_2}{dt}$$

$$T_{k2} = K_2\theta_2 \quad ; \quad T_{k1} = K_1(\theta_2 - \theta_1)$$

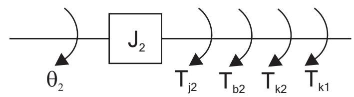


Fig 3.

By Newton's second law, $T_{j2} + T_{b2} + T_{k2} + T_{k1} = 0$

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2\theta_2 + K_1(\theta_2 - \theta_1) \qquad .....(2)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system we get,

$$\left(\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} \; ; \; \frac{d\theta}{dt} = \omega \text{ and } \theta = \int \omega \, dt \right)$$

$$J_1 \frac{d\omega_1}{dt} + B_1\omega_1 + K_1 \int (\omega_1 - \omega_2)\, dt = T \qquad .....(3)$$

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + K_2 \int \omega_2 dt + K_1 \int (\omega_2 - \omega_1)\, dt = 0 \qquad .....(4)$$

TORQUE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (J_1 and J_2). Hence the torque-voltage analogous electrical circuit will have two meshes. The torque applied to J_1 is represented by a voltage source in first mesh. The elements J_1, B_1 and K_1 are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements J_2, B_2, K_2 and K_1 are connected to second node. Hence they are represented by analogous elements in mesh-2 forming a closed path.

The element K_1 is common between node-1 and 2 and so it is represented by analogous element as common element between two meshes. The torque-voltage electrical analogous circuit is shown in fig 4.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$\omega_1 \rightarrow i_1$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$\omega_2 \rightarrow i_2$			

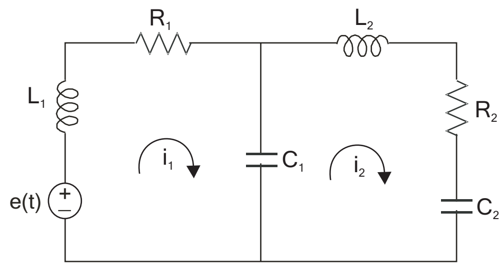


Fig 4 : *Torque-voltage electrical analogous circuit.*

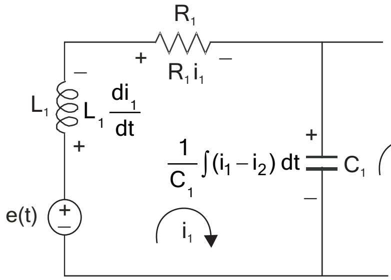


Fig 5 : *Mesh-1 of analogous circuit.*

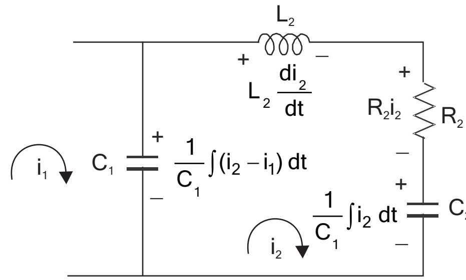


Fig 6 : *Mesh-2 of analogous circuit..*

The mesh basis equations using Kirchoff 's voltage law for the circuit shown in fig 4 are given below (Refer fig 5 and 6).

$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int(i_1-i_2)=e(t) \quad .....(5)$$

$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2dt+\frac{1}{C_2}\int(i_2-i_1)\,dt=0 \quad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

TORQUE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (J_1 and J_2). Hence the torque-current analogous electrical circuit will have two nodes. The torque applied to J_1 is represented as a current source connected to node-1 in analogous electrical circuit.

The elements J_1, B_1 and K_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements J_2, B_2, K_2 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit.

The element K_1 is common between node-1 and 2. So it is represented by analogous element as common element between node-1 and 2. The torque-current electrical analogous circuit is shown in fig 7.

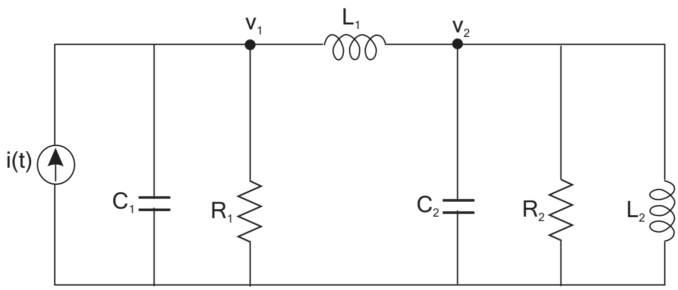


Fig 7 : *Torque-current electrical analogous circuit.*

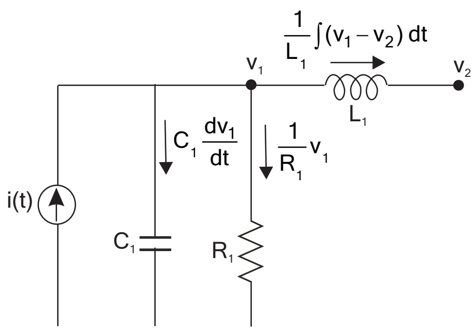


Fig 8 : *Node-1 of analogous circuit.*

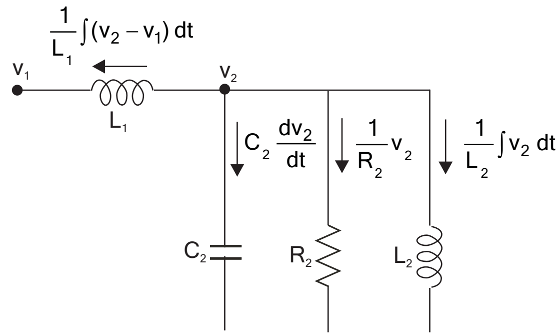


Fig 9 : *Node-2 of analogous circuit.*

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$ $\quad B_1 \rightarrow 1/R_1$ $\quad \omega_1 \rightarrow v_1$ $\quad J_1 \rightarrow C_1$ $\quad K_1 \rightarrow 1/L_1$

$B_2 \rightarrow 1/R_2$ $\quad \omega_2 \rightarrow v_2$ $\quad J_2 \rightarrow C_2$ $\quad K_2 \rightarrow 1/L_2$

The node basis equations using Kirchoff's current law for the circuit shown in fig 7 are given below (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}v_1 + \frac{1}{L_1}\int (v_1 - v_2)\,dt = i(t) \qquad .....(7)$$

$$C_1\frac{dv_2}{dt} + \frac{1}{R_2}v_2 + \frac{1}{L_2}\int v_2\,dt + \frac{1}{L_1}\int (v_2 - v_1)\,dt = i(t) \qquad .....(8)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

EXAMPLE 1.13

Write the differential equations governing the mechanical rotational system shown in fig 1. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equations.

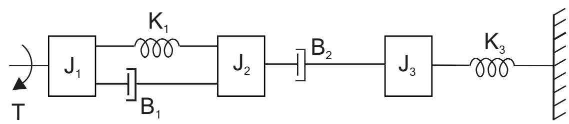


Fig 1.

SOLUTION

The given mechanical rotational system has three nodes (moment of inertia of masses). The differential equations governing the mechanical rotational system are given by torque balance equations at these nodes.

Let the angular displacements of J_1, J_2 and J_3 be θ_1, θ_2 and θ_3 respectively. The corresponding angular velocities be ω_1, ω_2 and ω_3.

The free body diagram of J_1 is shown in fig 2. The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

$$T_{j1} = J_1\frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b1} = B_1\frac{d(\theta_1 - \theta_2)}{dt}$$

$$T_{k1} = K_1(\theta_1 - \theta_2)$$

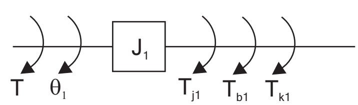


Fig 2.

By Newton's second law, $T_{j1} + T_{b1} + T_{k1} = T$

$$J_1\frac{d^2\theta_1}{dt^2} + B_1\frac{d(\theta_1 - \theta_2)}{dt} + K_1(\theta_1 - \theta_2) = T \qquad .....(1)$$

The free body diagram of J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b2}, T_{b1} and T_{k1}.

$$T_{j2} = J_2\frac{d^2\theta_2}{dt^2} \quad ; \quad T_{b2} = B_2\frac{d(\theta_2 - \theta_3)}{dt}$$

$$T_{k1} = K_1(\theta_2 - \theta_1) \quad ; \quad T_{b1} = B_1 \frac{d(\theta_2 - \theta_1)}{dt}$$

By Newton's second law, $T_{j2} + T_{b2} + T_{b1} + T_{k1} = 0$

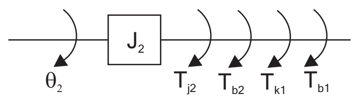


Fig 3.

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d(\theta_2 - \theta_3)}{dt} + B_1 \frac{d(\theta_2 - \theta_1)}{dt} + K_1(\theta_2 - \theta_1) = 0 \qquad .....(2)$$

The free body diagram of J_3 is shown in fig 4. The opposing torques are marked as T_{j3}, T_{b2}, and T_{k3}.

$$T_{j3} = J_3 \frac{d^2\theta_3}{dt^2} \quad ; \quad T_{b2} = B_2 \frac{d(\theta_3 - \theta_2)}{dt} \quad ; \quad T_{k3} = K_3\theta_3$$

By Newton's second law, $T_{j3} + T_{b2} + T_{k3} = 0$

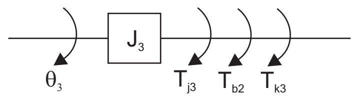


Fig 4.

$$\therefore \; J_3 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d(\theta_3 - \theta_2)}{dt} + K_3\theta_3 = 0 \qquad .....(3)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system we get,

$$\left(\text{i.e.,} \; \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} \quad ; \quad \frac{d\theta}{dt} = \omega \quad \text{and} \quad \theta = \int \omega \, dt\right)$$

$$J_1 \frac{d\omega_1}{dt} + B_1(\omega_1 - \omega_2) + K_1 \int (\omega_1 - \omega_2) \, dt = T \qquad .....(4)$$

$$J_2 \frac{d\omega_2}{dt} + B_1(\omega_2 - \omega_1) + B_2(\omega_2 - \omega_3) + K_1 \int (\omega_2 - \omega_1) \, dt = 0 \qquad .....(5)$$

$$J_3 \frac{d\omega_3}{dt} + B_2(\omega_3 - \omega_2) + K_3 \int \omega_3 dt = 0 \qquad .....(6)$$

TORQUE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (J_1, J_2 and J_3). Hence the torque-voltage analogous electrical circuit will have three meshes. The torque applied to J_1 is represented by a voltage source in first mesh.

The elements J_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements J_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous element in mesh-2 forming a closed path. The element J_3, B_2 and K_3 are connected to third node. Hence they are represented by analogous element in mesh-3 forming a closed path.

The elements K_1 and B_1 are common between the nodes-1 and 2 and so they are represented by analogous element as common between mesh-1 and 2. The element B_2 is common between the nodes-2 and 3 and so it is represented by analogous element as common element between the mesh-2 and 3. The torque-voltage electrical analogous circuit is shown in fig 5.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$	$\omega_1 \rightarrow i_1$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
	$\omega_2 \rightarrow i_2$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_3 \rightarrow 1/C_3$
	$\omega_3 \rightarrow i_3$	$J_3 \rightarrow L_3$		

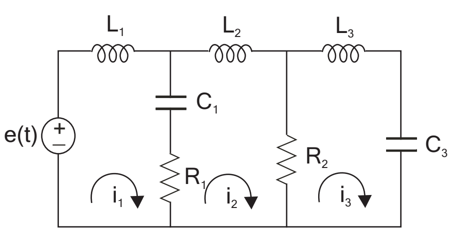


Fig 5 : *Torque-voltage electrical analogous circuit.*

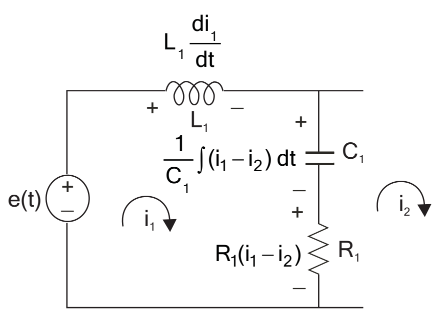


Fig 6 : *Mesh-1 of analogous circuit.*

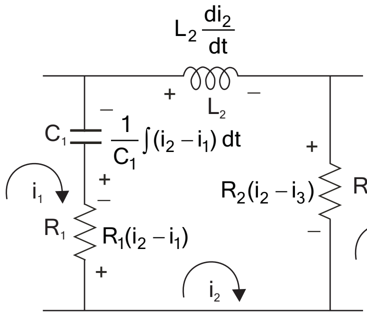


Fig 7 : Mesh-2 of analogous circuit.

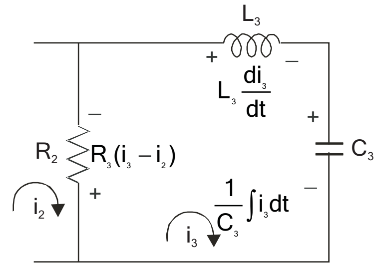


Fig 8 : Mesh-3 of analogous circuit.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 5 are given below (Refer fig 6, 7 and 8).

$$L_1\frac{di_1}{dt}+R_1(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)\,dt=e(t) \qquad .....(7)$$

$$L_2\frac{di_2}{dt}+R_1(i_2-i_1)+R_2(i_2-i_3)+\frac{1}{C_1}\int(i_2-i_1)\,dt=0 \qquad .....(8)$$

$$L_3\frac{di_3}{dt}+R_2(i_3-i_2)+\frac{1}{C_3}\int i_3\,dt=0 \qquad .....(9)$$

It is observed that the mesh basis equations (7), (8) and (9) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

TORQUE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (J_1, J_2 and J_3). Hence the torque-current analogous electrical circuit will have three nodes. The torque applied to J_1 is represented as a current source connected to node-1 in analogous electrical circuit.

The elements K_1, J_1 and B_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements J_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit. The elements J_3, B_2, and K_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node-3 in analogous electrical circuit.

The elements K_1 and B_1 are common between node-1 and 2 and so they are represented by analogous element as common elements between node-1 and 2. The element B_2 is common between node-2 and 3 and so it is represented as common element between node-2 and 3 in analogous circuit. The torque-current electrical analogous circuit is shown in fig 9.

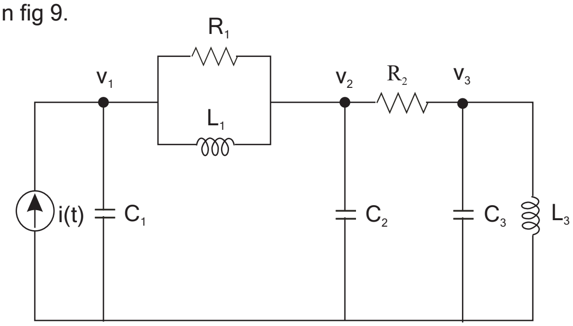


Fig 9 : Torque-current electrical analogous circuit.

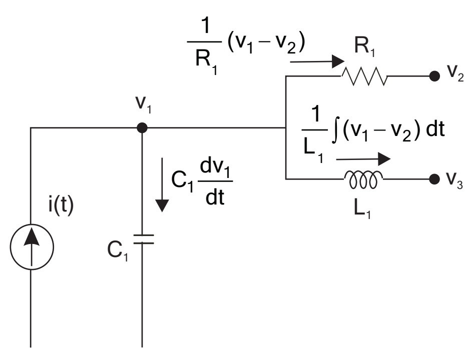


Fig 10 : Node-1 of analogous circuit.

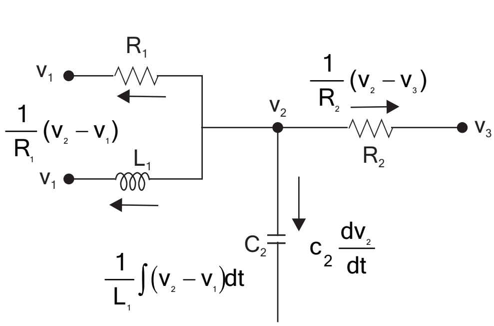


Fig 11 : Node-2 of analogous circuit.

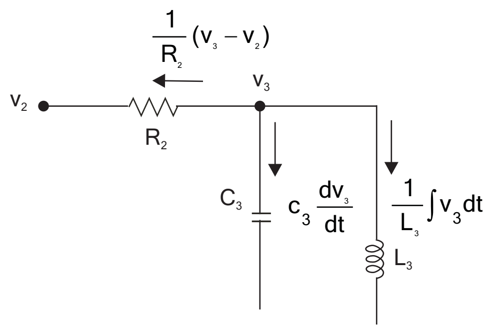


Fig 12 : Node-3 of analogous circuit.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$	$\omega_1 \rightarrow v_1$	$J_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
	$\omega_2 \rightarrow v_2$	$J_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_3 \rightarrow 1/L_3$
	$\omega_3 \rightarrow v_3$	$J_3 \rightarrow C_3$		

The node basis equations using Kirchoff's current law for the circuit shown in fig 9 are given below (Refer fig 10, 11 and 12).

$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}(v_1 - v_2) + \frac{1}{L_1}\int (v_1 - v_2)\,dt = i(t) \qquad .....(10)$$

$$C_2\frac{dv_2}{dt} + \frac{1}{R_1}(v_2 - v_1) + \frac{1}{R_2}(v_2 - v_3) + \frac{1}{L_1}\int (v_2 - v_1)\,dt = 0 \qquad .....(11)$$

$$C_3\frac{dv_3}{dt} + \frac{1}{R_2}(v_3 - v_2) + \frac{1}{L_3}\int v_3 dt = 0 \qquad .....(12)$$

It is observed that the node basis equations (10), (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

1.11 BLOCK DIAGRAMS

A control system may consist of a number of components. In control engineering to show the functions performed by each component, we commonly use a diagram called the block diagram. A ***block diagram*** of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components. The elements of a block diagram are ***block***, ***branch point*** and ***summing point.***

BLOCK

In a block diagram all system variables are linked to each other through functional blocks. The ***functional block*** or simply ***block*** is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Figure 1.25 shows the block diagram of functional block.

The arrowhead pointing towards the block indicates the input, and the arrowhead leading away from the block represents the output. Such arrows are referred to as signals. The output signal from the block isgiven by the product of input signal and transfer function in the block.

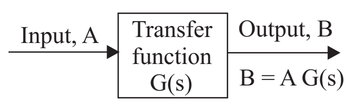


Fig 1.25 : Functional block.

SUMMING POINT

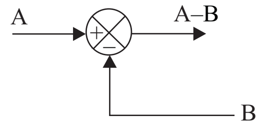


Fig 1.26 : *Summing point.*

Summing points are used to add two or more signals in the system. Referring to figure 1.26, a circle with a cross is the symbol that indicates a summing operation.

The plus or minus sign at each arrowhead indicates whether the signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

BRANCH POINT

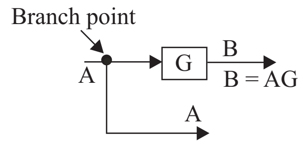


Fig 1.27 : *Branch point.*

A ***branch point*** is a point from which the signal from a block goes concurrently to other blocks or summing points.

CONSTRUCTING BLOCK DIAGRAM FOR CONTROL SYSTEMS

A control system can be represented diagramatically by block diagram. The differential equations governing the system are used to construct the block diagram. By taking Laplace transform the differential equations are converted to algebraic equations. The equations will have variables and constants. From the working knowledge of the system the input and output variables are identified and the block diagram for each equation can be drawn. Each equation gives one section of block diagram. The output of one section will be input for another section. The various sections are interconnected to obtain the overall block diagram of the system.

EXAMPLE 1.14

Construct the block diagram of armature controlled dc motor.

SOLUTION

The differential equations governing the armature controlled dc motor are (refer section 1.7),

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \qquad .....(1)$$

$$T = K_t i_a \qquad .....(2)$$

$$T = J\frac{d\omega}{dt} + B\omega \qquad .....(3)$$

$$e_b = K_b \omega \qquad .....(4)$$

$$\omega = \frac{d\theta}{dt} \qquad .....(5)$$

On taking Laplace transform of equation (1) we get,

$$V_a(s) = I_a(s)R_a + L_a s\, I_a(s) + E_b(s) \qquad .....(6)$$

In equation (6), $V_a(s)$ and $E_b(s)$ are inputs and $I_a(s)$ is the output. Hence the equation (6) is rearranged and the block diagram for this equation is shown in fig 1.

$$V_a(s) - E_b(s) = I_a(s)\,[R_a + s\,L_a]$$

$$\therefore\ I_a(s) = \frac{1}{R_a + s\,L_a}[V_a(s) - E_b(s)]$$

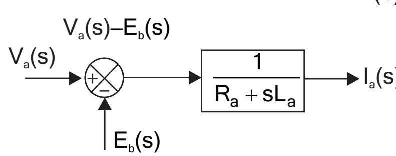


Fig 1.

On taking Laplace transform of equation (2) we get,

$$T(s) = K_t\, I_a(s) \qquad .....(7)$$

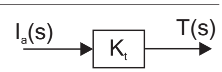


Fig 2.

In equation (7), $I_a(s)$ is the input and $T(s)$ is the output. The block diagram for this equation is shown in fig 2.

On taking Laplace transform of equation (3) we get,

$$T(s) = Js\,\omega(s) + B\,\omega(s) \qquad .....(8)$$

In equation (8), T(s) is the input and ω(s) is the output. Hence the equation (8) is rearranged and the block diagram for this equation is shown in fig (3).

$$T(s) = (Js + B)\,\omega(s)$$

$$\therefore\ \omega(s) = \frac{1}{Js+B}\,T(s)$$

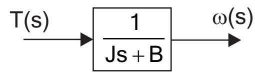


Fig 3.

On taking Laplace transform of equation (4) we get,

$$E_b(s) = K_b\,\omega(s) \qquad .....(9)$$

In equation (9), ω(s) is the input and $E_b(s)$ is the output. The block diagram for this equation is shown in fig 4.

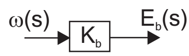


Fig 4.

On taking Laplace transform of equation (5) we get,

$$\omega(s) = s\,\theta(s) \qquad .....(10)$$

In equation (10), ω(s) is the input and θ(s) is the output. Hence equation (10) is rearranged and the block diagram for this equation is shown in fig 5.

$$\theta(s) = \frac{1}{s}\,\omega(s)$$

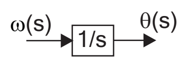


Fig 5.

The overall block diagram of armature controlled dc motor is obtained by connecting the various sections shown in fig 1 to fig 5. The overall block diagram is shown in fig 6.

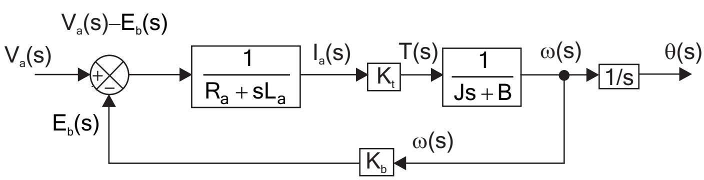


Fig 6 : *Block diagram of armature controlled dc motor.*

EXAMPLE 1.15

Construct the block diagram of field controlled dc motor.

SOLUTION

The differential equations governing the field controlled dc motor are (refer section 1.8),

$$v_f = R_f i_f + L_f \frac{di_f}{dt} \qquad .....(1)$$

$$T = K_{tf} i_f \qquad .....(2)$$

$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \qquad .....(3)$$

On taking Laplace transform of equation (1) we get,

$$V_f(s) = R_f I_f(s) + L_f s\, I_f(s) \qquad .....(4)$$

In equation (4), $V_f(s)$ is the input and $I_f(s)$ is the output. Hence the equation (4) is rearranged and the block diagram for this equation is shown in fig 1.

$$V_f(s) = I_f(s)\,[R_f + sL_f]$$

$$\therefore\; I_f(s) = \frac{1}{R_f + sL_f} V_f(s)$$

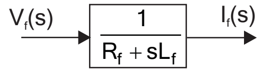


Fig 1.

On taking Laplace transform of equation (2) we get,

$$T(s) = K_{tf} I_f(s) \qquad .....(5)$$

In equation (5), $I_f(s)$ is the input and T(s) is the output. The block diagram for this equation is shown in fig 2.

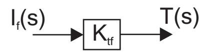


Fig 2.

On taking Laplace transform of equation (3) we get,

$$T(s) = J\,s^2\,\theta(s) + B\,s\,\theta(s) \qquad .....(6)$$

In equation (6), T(s) is input and $\theta(s)$ is the output. Hence equation (6) is rearranged and the block diagram for this equation is shown in fig 3.

$$T(s) = (Js^2 + Bs)\,\theta(s)$$

$$\therefore\; \theta(s) = \frac{1}{Js^2 + Bs} T(s)$$

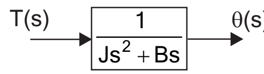


Fig 3.

The overall block diagram of field controlled dc motor is obtained by connecting the various section shown in fig 1 to fig 3. The overall block diagram is shown in fig 4.

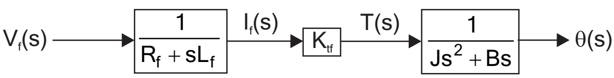


Fig 4 : *Block diagram of field controlled dc motor.*

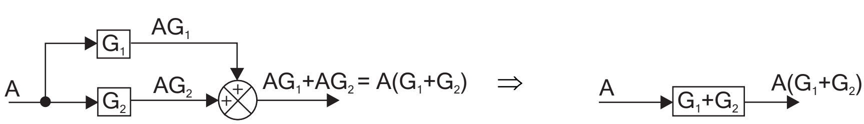
BLOCK DIAGRAM REDUCTION

The block diagram can be reduced to find the overall transfer function of the system. The following rules can be used for block diagram reduction . The rules are framed such that any modification made on the diagram does not alter the input-output relation.

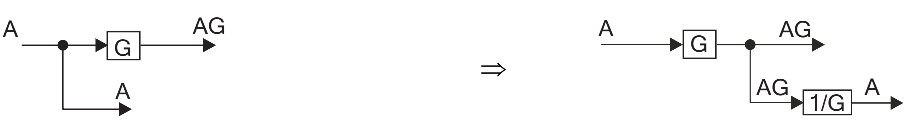
RULES OF BLOCK DIAGRAM ALGEBRA

Rule-1 : ***Combining the blocks in cascade***

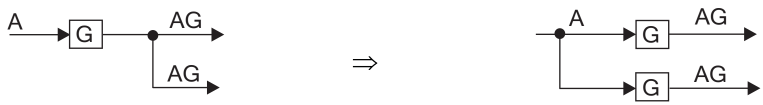
Rule-2 : ***Combining Parallel blocks (or combining feed forward paths)***



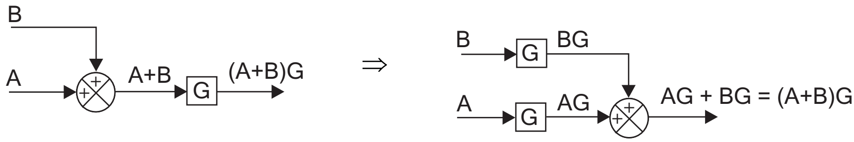
Rule-3 : ***Moving the branch point ahead of the block***



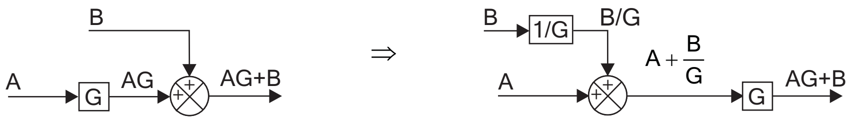
Rule-4 : ***Moving the branch point before the block***



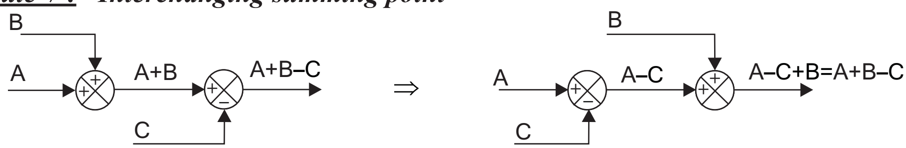
Rule-5 : ***Moving the summing point ahead of the block***



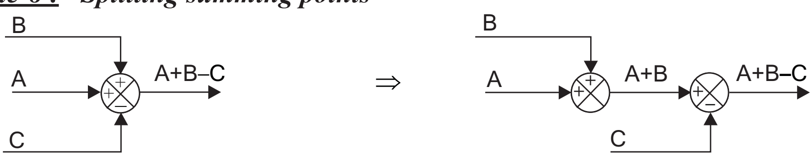
Rule-6 : ***Moving the summing point before the block***



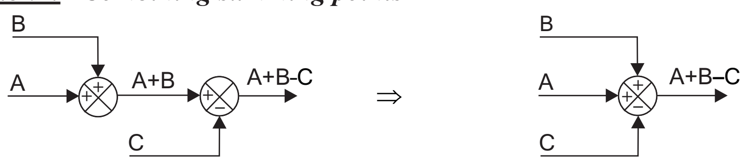
Rule-7 : ***Interchanging summing point***



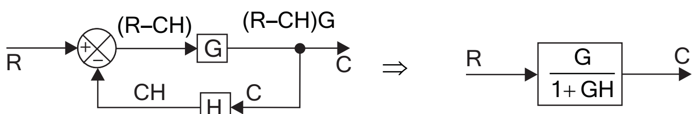
Rule-8 : ***Splitting summing points***



Rule-9 : ***Combining summing points***



Rule-10 : ***Elimination of (negative) feedback loop***

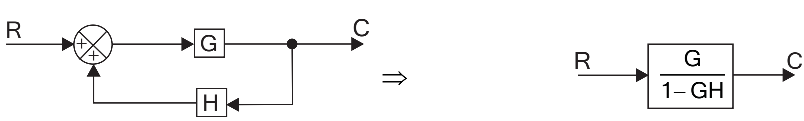


Proof :

$$C = (R - CH)G \quad \Rightarrow \quad C = RG - CHG \quad \Rightarrow \quad C + CHG = RG$$

$$C(1 + HG) = RG \quad \Rightarrow \quad \frac{C}{R} = \frac{G}{1 + GH}$$

Rule-11 : ***Elimination of (positive) feedback loop***



EXAMPLE 1.16

Reduce the block diagram shown in fig 1 and find C/R.

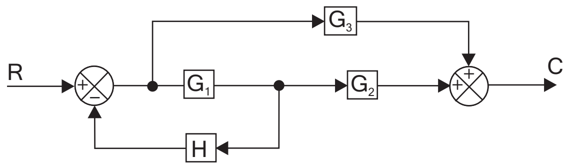
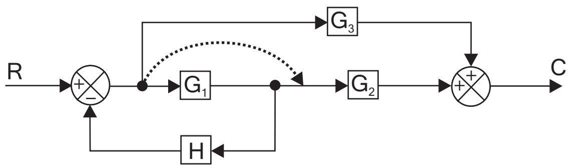


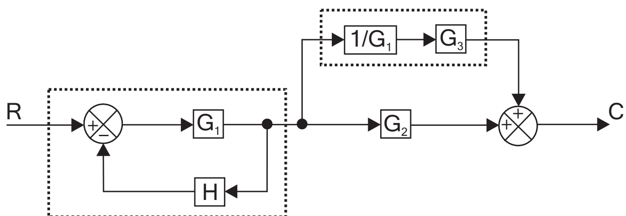
Fig 1.

SOLUTION

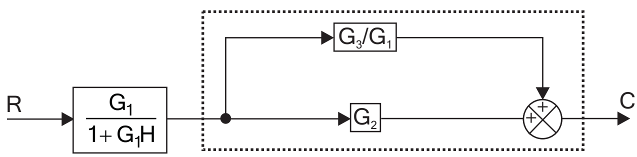
Step 1: Move the branch point after the block.



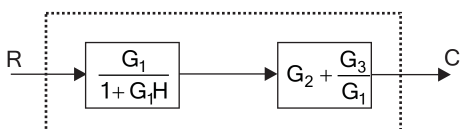
Step 2: Eliminate the feedback path and combining blocks in cascade.



Step 3: Combining parallel blocks



Step 4: Combining blocks in cascade



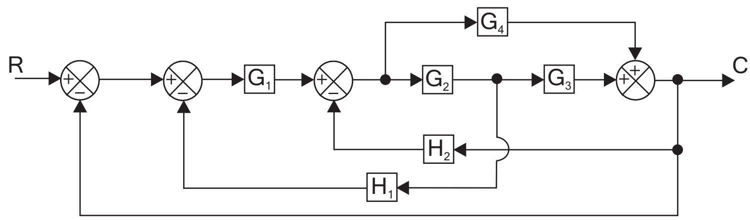
$$\frac{C}{R}=\left(\frac{G_1}{1+G_1H}\right)\left(G_2+\frac{G_3}{G_1}\right)=\left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_1G_2+G_3}{G_1}\right)=\frac{G_1G_2+G_3}{1+G_1H}$$

RESULT

The overall transfer function of the system, $\frac{C}{R}=\frac{G_1G_2+G_3}{1+G_1H}$

EXAMPLE 1.17

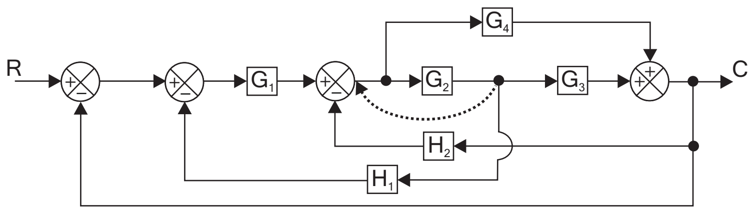
Using block diagram reduction technique find closed loop transfer function of the system whose block diagram is shown in fig 1.



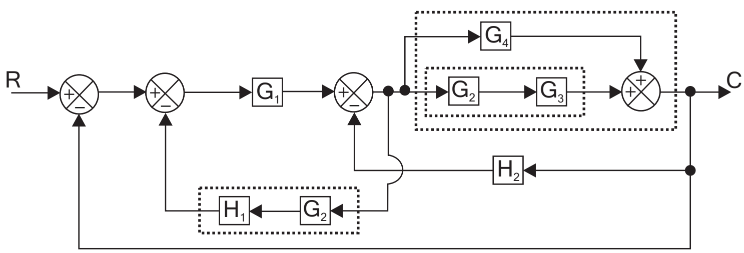
SOLUTION

Fig 1.

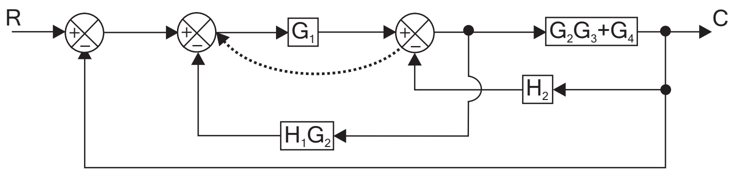
Step 1: Moving the branch point before the block



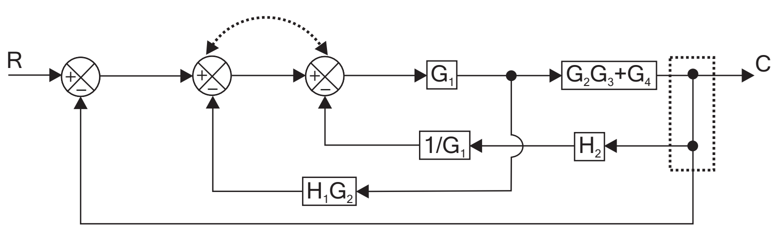
Step 2: Combining the blocks in cascade and eliminating parallel blocks



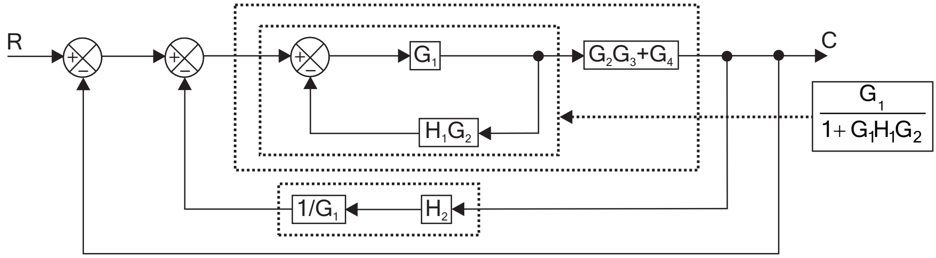
Step 3: Moving summing point before the block.



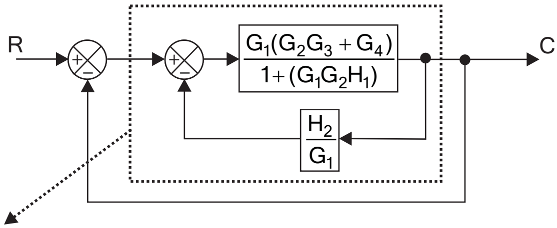
Step 4: Interchanging summing points and modifying branch points.



Step 5: Eliminating the feedback path and combining blocks in cascade

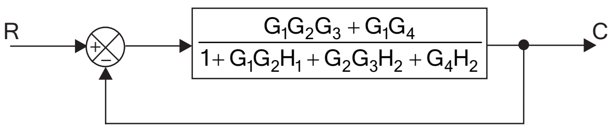


Step 6: Eliminating the feedback path



$$\frac{\dfrac{G_1(G_2G_3+G_4)}{1+G_2G_1H_1}}{1+\dfrac{G_1(G_2G_3+G_4)}{1+G_2G_1H_1}\,\dfrac{H_2}{G_1}} \Rightarrow \frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_2G_1H_1}}{\dfrac{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}{1+G_1G_2H_1}} \Rightarrow \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}$$

Step 7: Eliminating the feedback path



$$\frac{C}{R}=\frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}{1+\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

RESULT

The overall transfer function is given by,

$$\frac{C}{R}=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_1G_2G_3+G_1G_4}$$

EXAMPLE 1.18

Determine the overall transfer function $\frac{C(s)}{R(s)}$ for the system shown in fig 1.

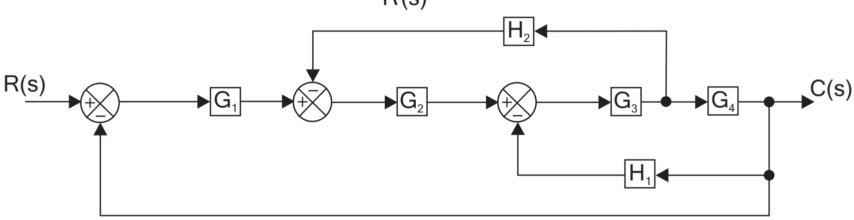
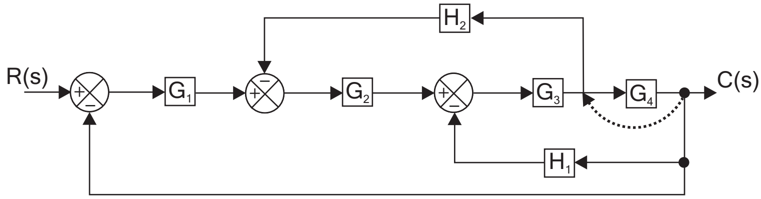


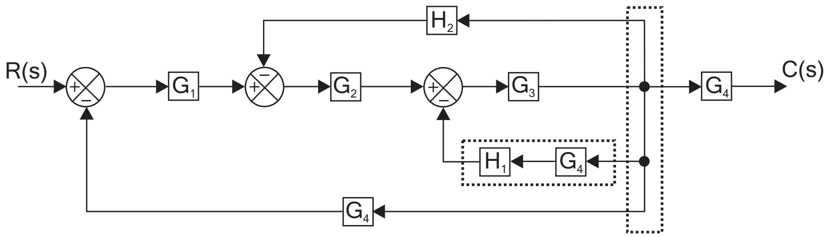
Fig 1.

SOLUTION

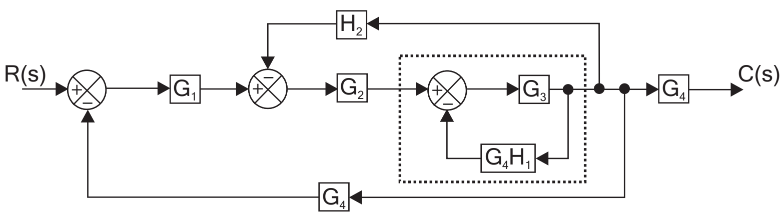
Step 1: Moving the branch point before the block



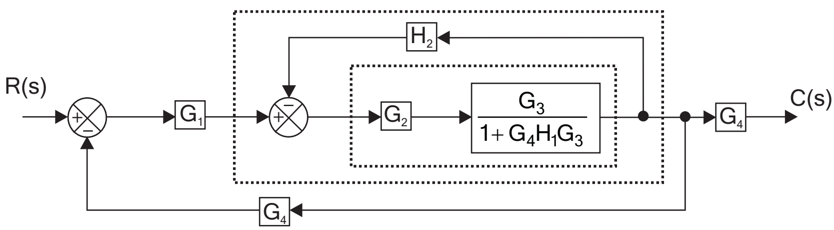
Step 2 : Combining the blocks in cascade and rearranging the branch points



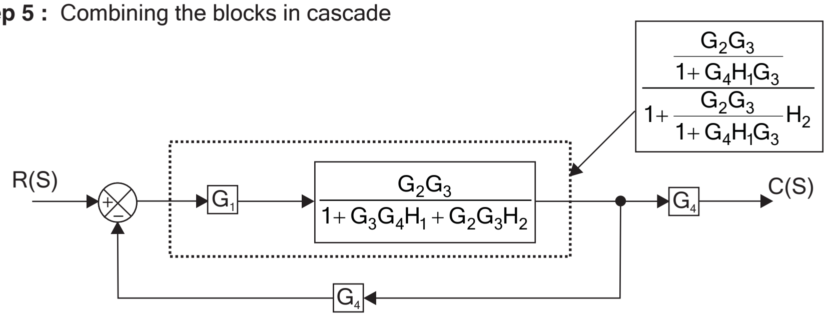
Step 3 : Eliminating the feedback path



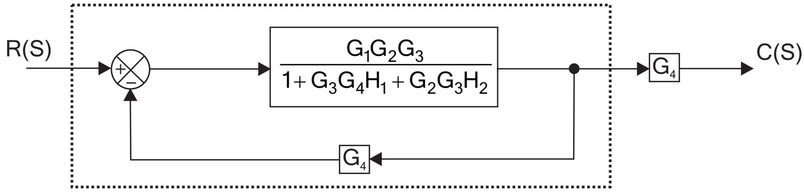
Step 4 : Combining the blocks in cascade and eliminating feedback path



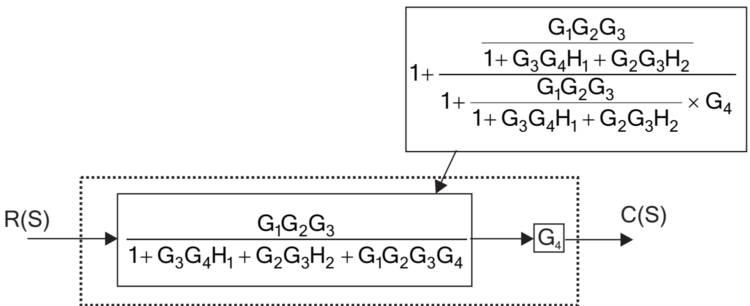
Step 5 : Combining the blocks in cascade



Step 6 : Eliminating the feedback path



Step 7 : Combining the blocks in cascade



RESULT

The overall transfer function of the system is given by,

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

EXAMPLE 1.19

For the system represented by the block diagram shown in fig 1. Evaluate the closed loop transfer function when the input R is (i) at station-I (ii) at station-II.

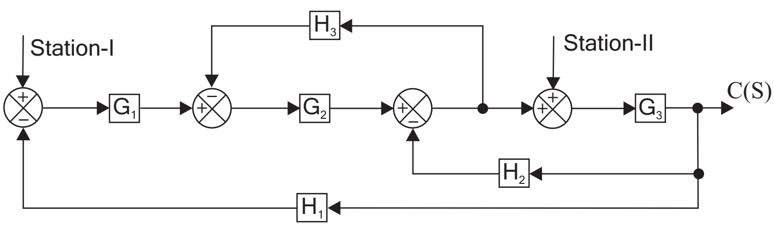


Fig 1.

SOLUTION

(i) Consider the input R is at station-I and so the input at station-II is made zero. Let the output be C^1. Since there is no input at station-II that summing point can be removed and resulting block diagram is shown in fig 2.

Step 1 : Shift the take off point of feedback H_3 beyond G_3 and rearrange the branch points

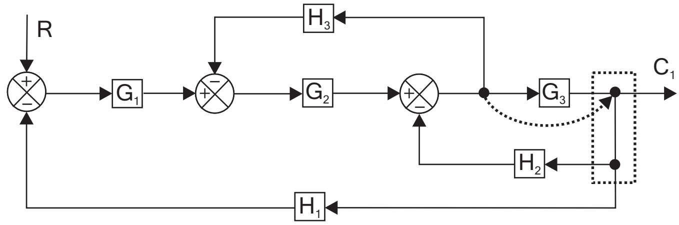
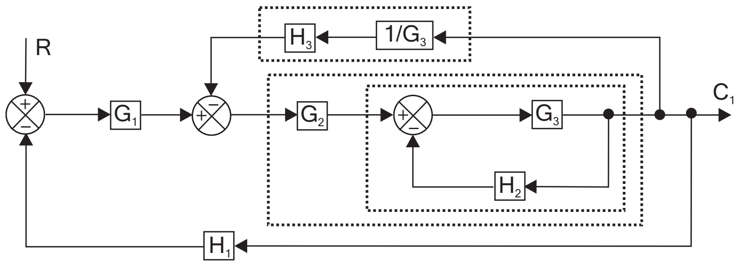
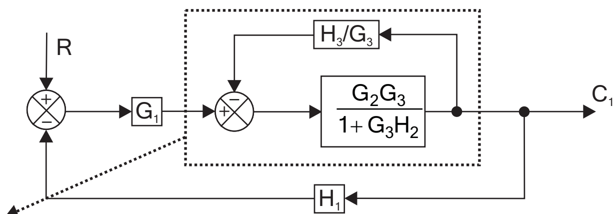


Fig 2.

Step 2 : Eliminating the feedback H_2 and combining blocks in cascade

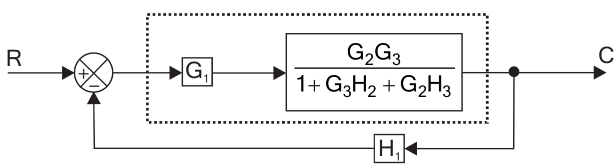


Step 3: Eliminating the feedback path

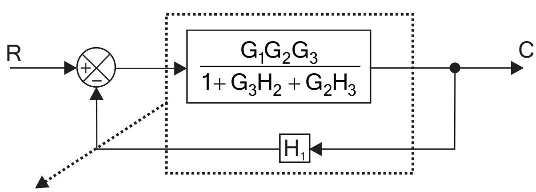


$$\frac{\dfrac{G_2G_3}{1+G_3H_2}}{1+\dfrac{G_2G_3}{1+G_3H_2}\times\dfrac{H_3}{G_3}} = \frac{\dfrac{G_2G_3}{1+G_3H_2}}{1+\dfrac{G_2H_3}{1+G_3H_2}} = \frac{G_2G_3}{1+G_3H_2+G_2H_3}$$

Step 4: Combining the blocks in cascade



Step 5: Eliminating feedback path H_1



$$\frac{\dfrac{G_1G_2G_3}{1+G_3H_2+G_2H_3}}{1+\dfrac{G_1G_2G_3}{1+G_3H_2+G_2H_3}\times H_1} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

$$\therefore \ \frac{C_1(s)}{R(s)}=\frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

(ii) Consider the input R at station-II, the input at station-I is made zero. Let output be C_2. Since there is no input in station-I that corresponding summing point can be removed and a negative sign can be attached to the feedback path gain H_1. The resulting block diagram is shown in fig 3.

Step 1: Combining the blocks in cascade, shifting the summing point of H_2 before G_2 and rearranging the branch points.

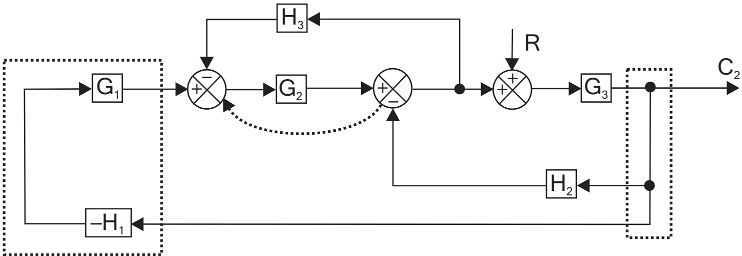
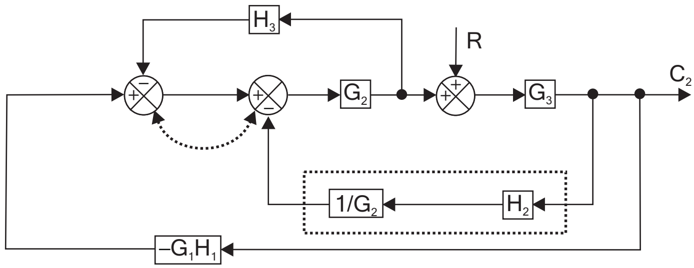
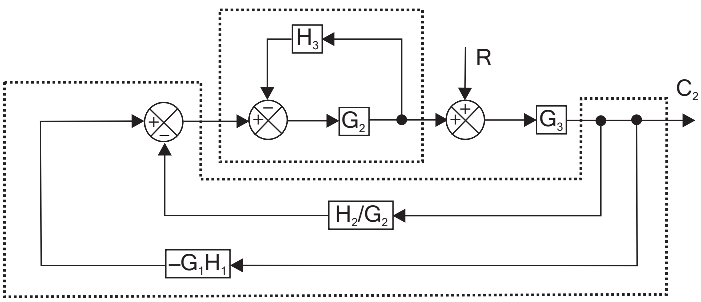


Fig 3.

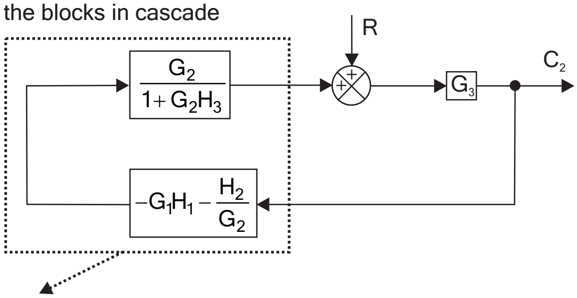
Step 2: Interchanging summing points and combining the blocks in cascade.



Step 3: Combining parallel blocks and eliminating feedback path

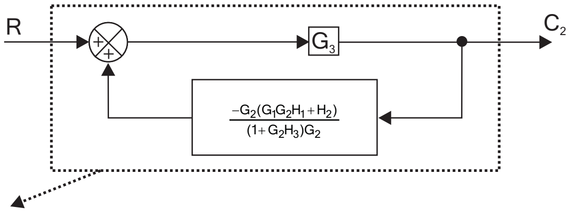


Step 4: Combining the blocks in cascade



$$\left(\frac{G_2}{1+G_2H_3}\right)\times\left(-G_1H_1-\frac{H_2}{G_2}\right)=\left(\frac{G_2}{1+G_2H_3}\right)\times\left(\frac{-G_1H_1G_2-H_2}{G_2}\right)=\frac{-G_2(G_1G_2H_1+H_2)}{(1+G_2H_3)G_2}$$

Step 5: Eliminating the feedback path



$$\frac{G_3}{1-\left(\frac{-(G_1G_2H_1+H_2)}{1+G_2H_3}\right)G_3} = \frac{G_3}{\frac{1+G_2H_3+G_3(G_1G_2H_1+H_2)}{1+G_2H_3}} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

$$\therefore \frac{C_2}{R} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

RESULT

The transfer function of the system with input at station-I is,

$$\frac{C_1}{R} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

The transfer function of the system with input at station-II is,

$$\frac{C_2}{R} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

EXAMPLE 1.20

For the system represented by the block diagram shown in the fig 1, determine and .

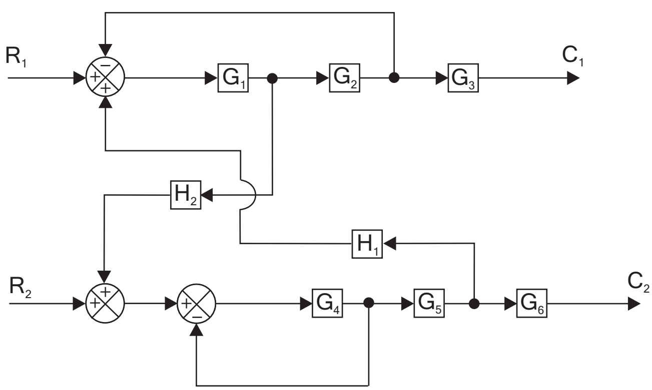


Fig 1

SOLUTION

Case (i) To find $\frac{C_1}{R_1}$

In this case set $R_2 = 0$ and consider only one output C_1. Hence we can remove the summing point which adds R_2 and need not consider G_6, since G_6 is on the open path. The resulting block diagram is shown in fig 2.

Step 1: Eliminating the feedback path

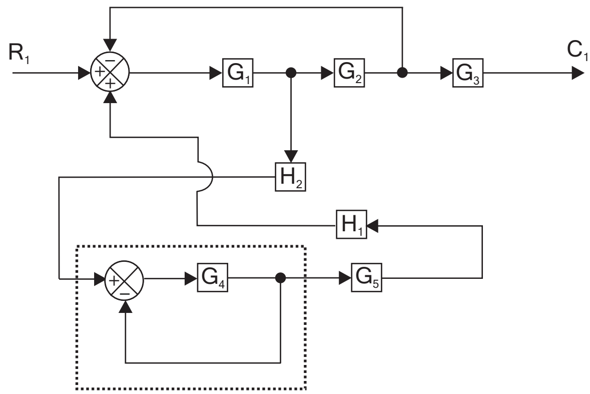
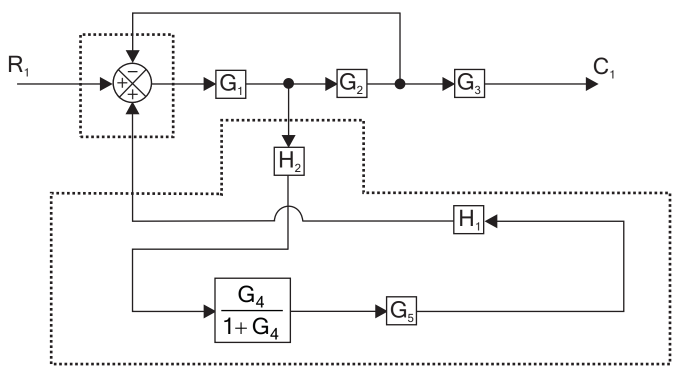
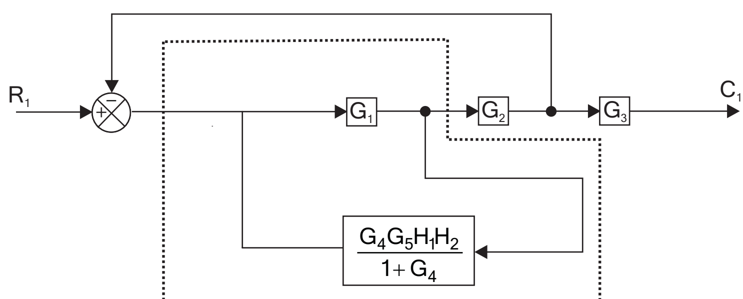


Fig 2.

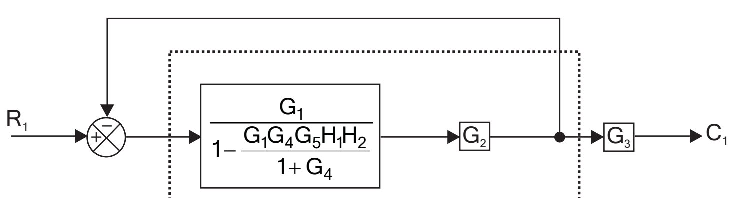
Step 2: Combining the blocks in cascade and splitting the summing point



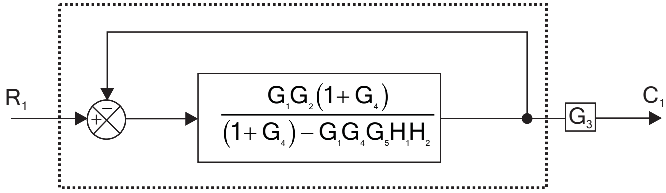
Step 3: Eliminating the feedback path

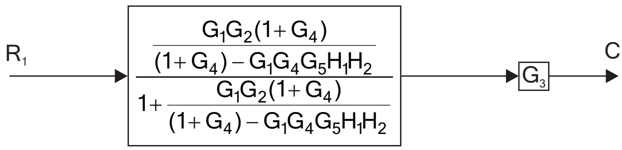


Step 4: Combining the blocks in cascade

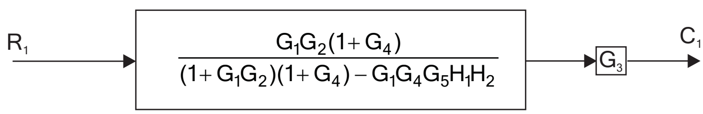


Step 5 : Eliminating the feedback path





Step 6: Combining the blocks in cascade



$$\frac{C_1}{R_1} = \frac{G_1G_2G_3(1+G_4)}{(1+G_1G_2)(1+G_4) - G_1G_4G_5H_1H_2}$$

Case 2 : To find $\frac{C_2}{R_1}$

In this case set R_2 = 0 and consider only one output C_2. Hence we can remove the summing point which adds R_2 and need not consider G_3, since G_3 is on the open path. The resulting block diagram is shown in fig 3.

Step 1: Eliminate the feedback path.

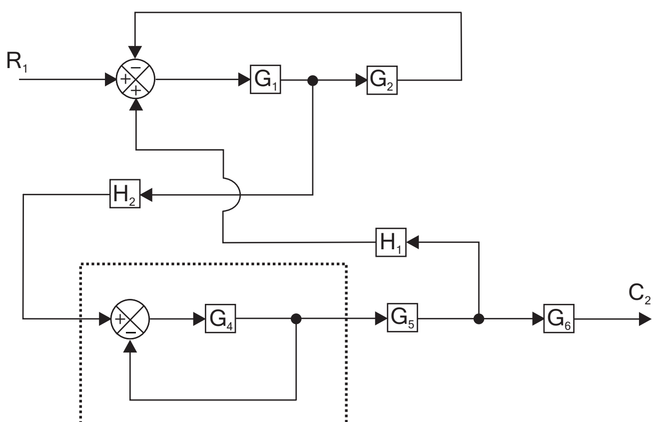
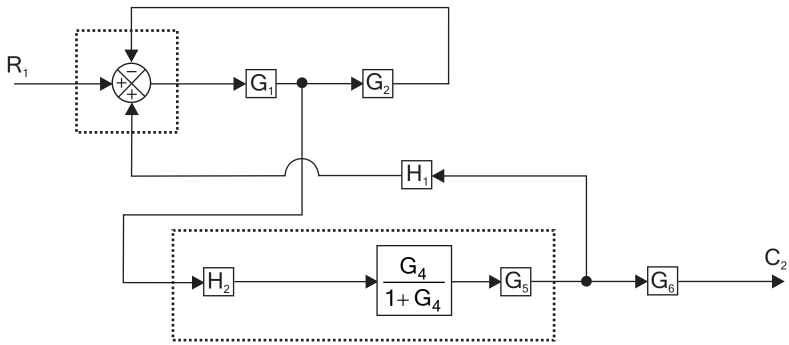
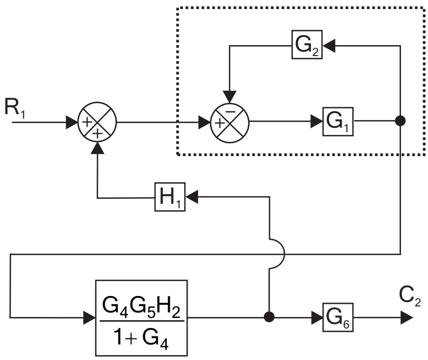


Fig 3.

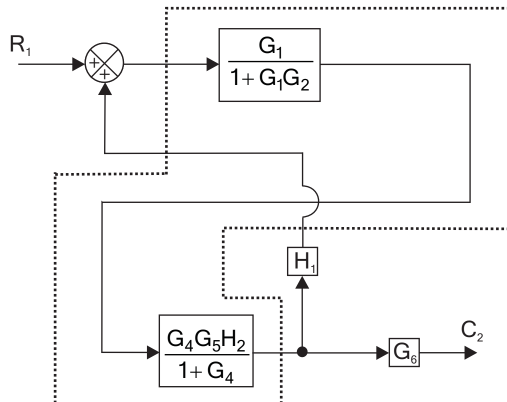
Step 2: Combining blocks in cascade and splitting the summing point



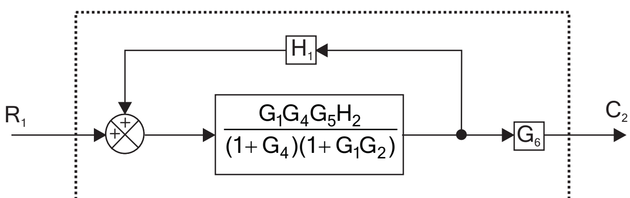
Step 3: Eliminating the feedback path

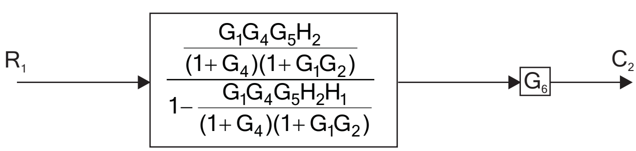


Step 4 : Combining the blocks in cascade

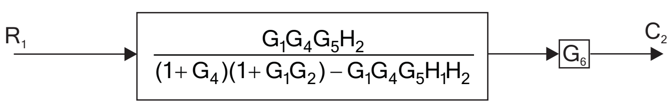


Step 5: Eliminating the feedback path





Step 6: Combining the blocks in cascade



$$\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6H_2}{(1+G_4)(1+G_1G_2) - G_1G_4G_5H_1H_2}$$

RESULT

The transfer function of the system when the input and output are R_1 and C_1 is given by,

$$\frac{C_1}{R_1} = \frac{G_1G_2G_3(1+G_4)}{(1+G_1G_2)(1+G_4) - G_1G_4G_5H_1H_2}$$

The transfer function of the system when the input and output are R_1 and C_2 is given by,

$$\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6H_2}{(1+G_4)(1+G_1G_2) - G_1G_4G_5H_1H_2}$$

EXAMPLE 1.21

Obtain the closed loop transfer function C(s)/R(s) of the system whose block diagram is shown in fig 1.

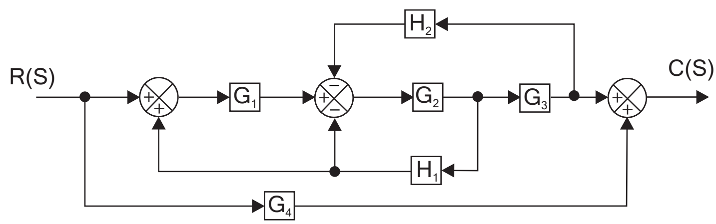
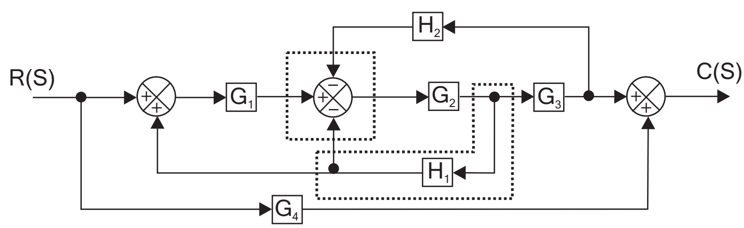


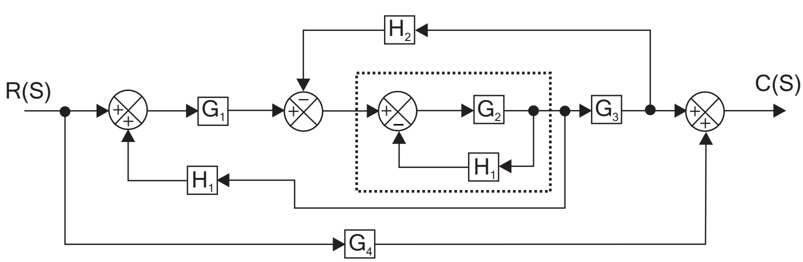
Fig 1.

SOLUTION

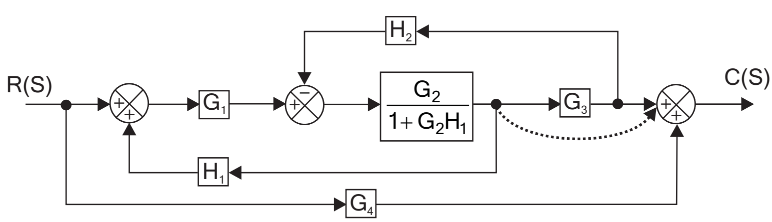
Step 1 : Splitting the summing point and rearranging the branch points



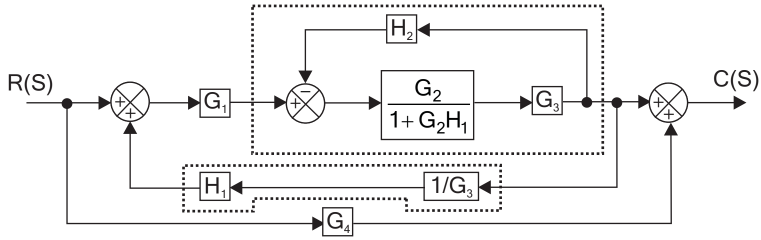
Step 2 : Eliminating the feedback path



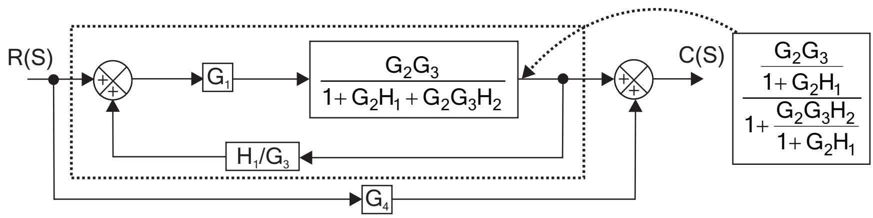
Step 3 : Shifting the branch point after the block.



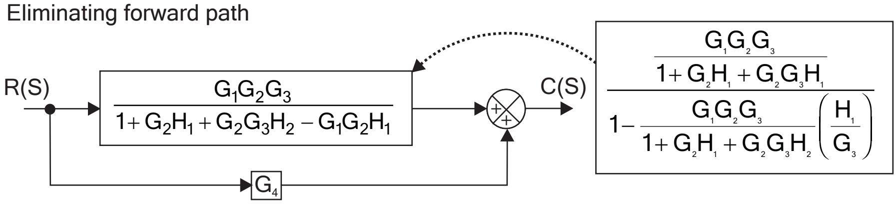
Step 4 : Combining the blocks in cascade and eliminating feedback path



Step 5 : Combining the blocks in cascade and eliminating feedback path



Step 6 : Eliminating forward path



$$\therefore \frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4$$

RESULT

The transfer function of the system is $\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4$

EXAMPLE 1.22

The block diagram of a closed loop system is shown in fig 1. Using the block diagram reduction technique determine the closed loop transfer function C(s)/R(s).

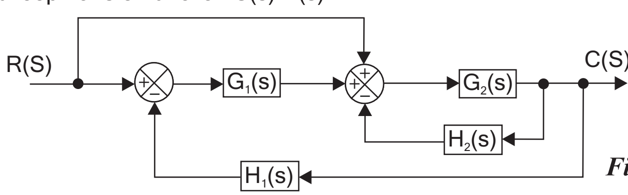
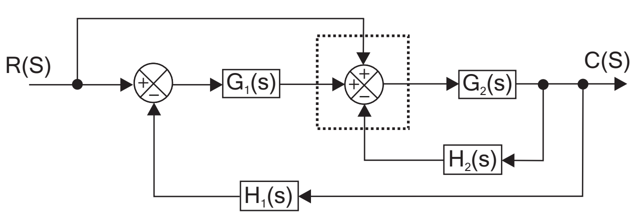


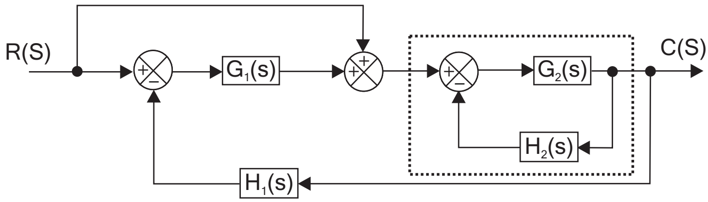
Fig 1.

SOLUTION

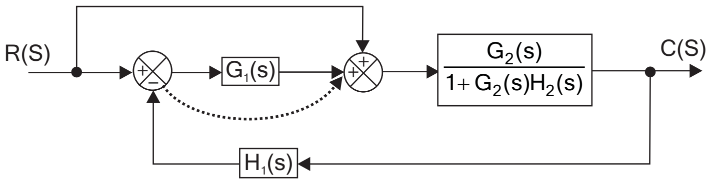
Step 1 : Splitting the summing point.



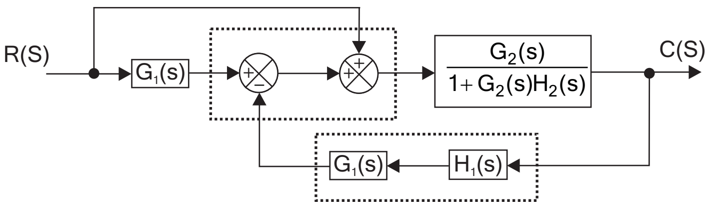
Step 2 : Eliminating the feedback path.



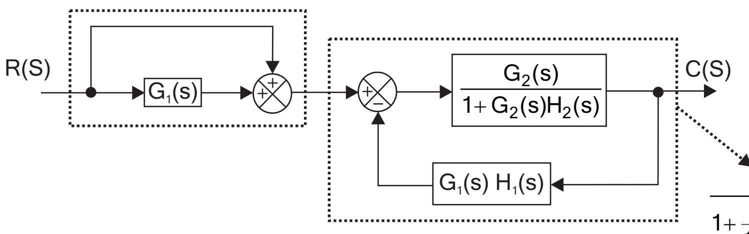
Step 3 : Moving the summing point after the block.



Step 4 : Interchanging the summing points and combining the blocks in cascade

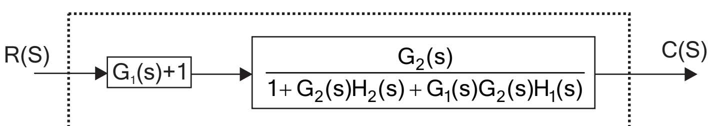


Step 5 : Eliminating the feedback path and feed forward path



$$\frac{\dfrac{G_2(s)}{1+G_2(s)H_2(s)}}{1+\dfrac{G_2(s)}{1+G_2(s)H_2(s)}G_1(s)H_1(s)}$$

Step 6 : Combining the blocks in cascade



$$\therefore \frac{C(s)}{R(s)} = \frac{G_2(s)\,[G_1(s)+1]}{1+G_2(s)\,H_2(s)+G_1(s)\,G_2(s)\,H_1(s)}$$

RESULT

The transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{G_2(s)\,[G_1(s)+1]}{1+G_2(s)\,H_2(s)+G_1(s)\,G_2(s)\,H_1(s)}$$

EXAMPLE 1.23

Using block diagram reduction technique find the transfer function C(s)/R(s) for the system shown in fig 1.

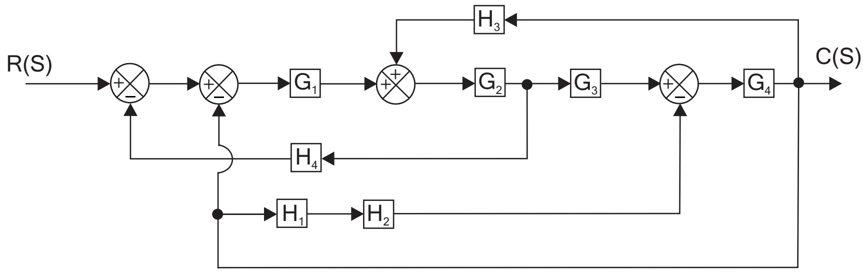
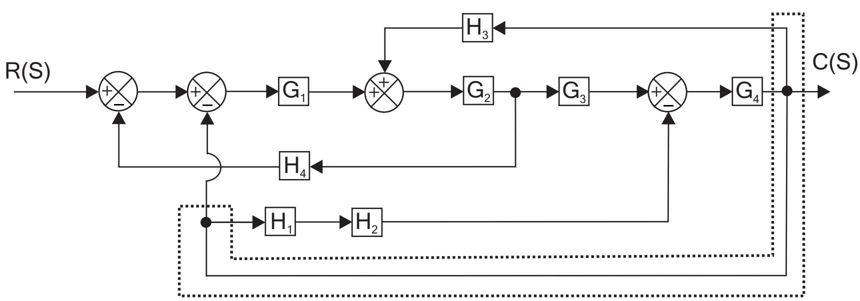


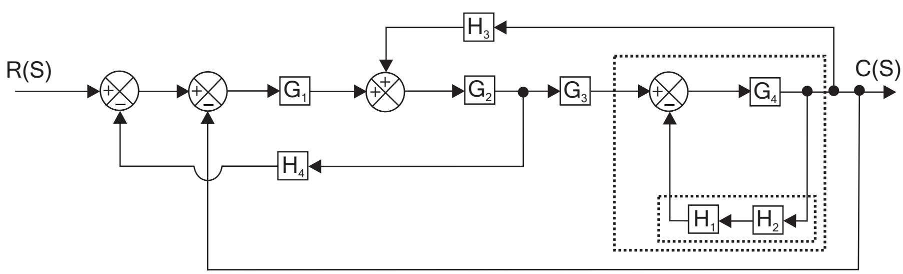
Fig 1.

SOLUTION

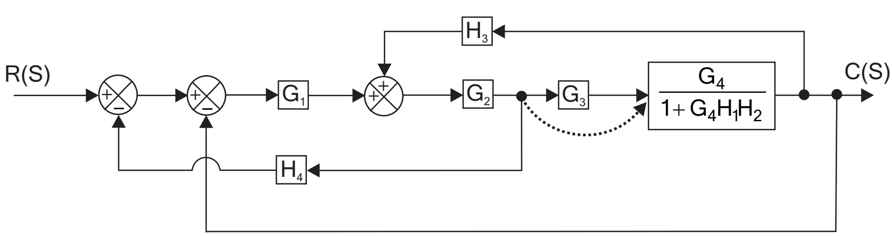
Step 1 : Rearranging the branch points



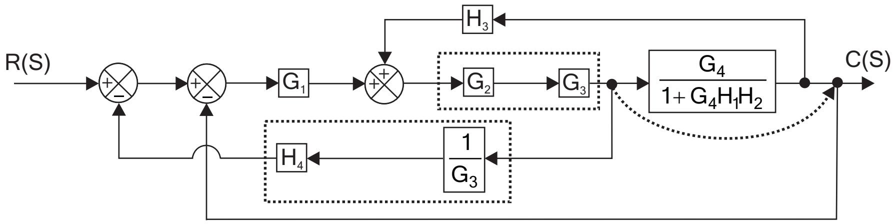
Step 2 : Combining the blocks in cascade and eliminating the feedback path.



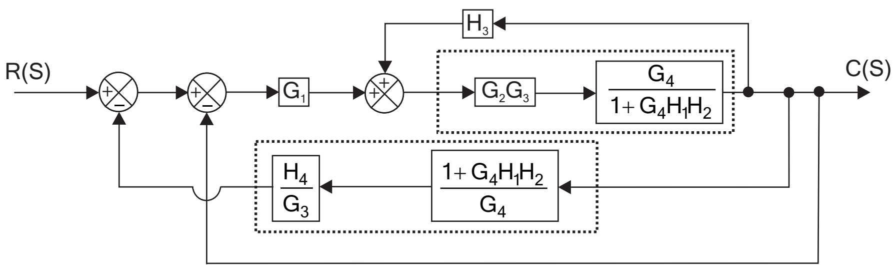
Step 3 : Moving the branch point after the block.



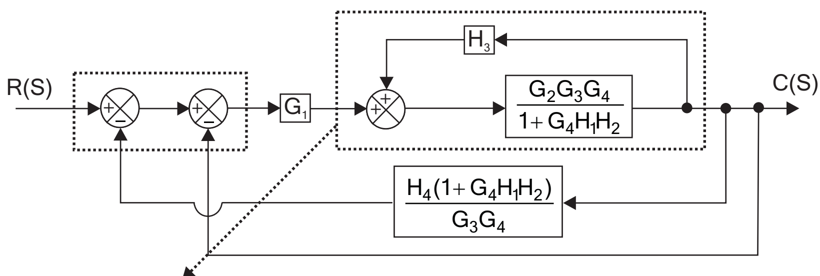
Step 4 : Moving the branch point and combining the blocks in cascade.



Step 5 : Combining the blocks in cascade

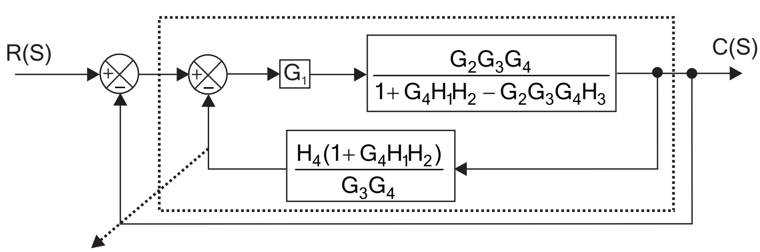


Step 6 : Eliminating feedback path and interchanging the summing points.



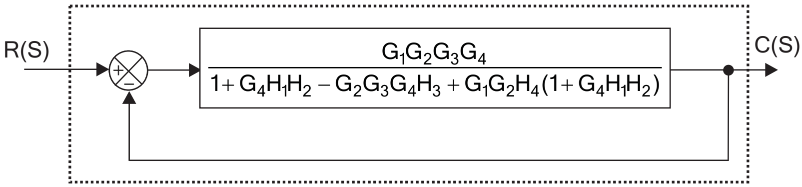
$$\frac{\dfrac{G_2G_3G_4}{1+G_4H_1H_2}}{1-\dfrac{G_2G_3G_4H_3}{1+G_4H_1H_2}} = \frac{G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}$$

Step 7 : Combining the blocks in cascade and eliminating the feedback path



$$\frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}}{1+\left(\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}\right)\left(\dfrac{H_4(1+G_4H_1H_2}{G_3G_4}\right)} = \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}$$

Step 8 : Eliminating the unity feedback path.



$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}{1+\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}$$

$$= \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)+G_1G_2G_3G_4}$$

$$= \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

RESULT

The transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

1.12 Signal flow graph

The signal flow graph is used to represent the control system graphically and it was developed by **S.J. Mason.**

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

It should be noted that the signal flow graph approach and the block diagram approach yield the same information. The advantage in signal flow graph method is that, using Mason's gain formula the overall gain of the system can be computed easily. This method is simpler than the tedious block diagram reduction techniques.

The signal flow graph depicts the flow of signals from one point of a system to another and gives the relationships among the signals. A signal flow graph consists of a network in which nodes are connected by directed branches. Each node represents a system variable and each branch connected between two nodes acts as a signal multiplier. Each branch has a gain or transmittance. When the signal pass through a branch, it gets multiplied by the gain of the branch.

In a signal flow graph, the signal flows in only one direction. The direction of signal flow is indicated by an arrow placed on the branch and the gain (multiplication factor) is indicated along the branch.

EXPLANATION OF TERMS USED IN SIGNAL FLOW GRAPH

Node : A node is a point representing a variable or signal.

Branch : A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance.

Transmittance : The gain acquired by the signal when it travels from one node to another is called transmittance. The transmittance can be real or complex.

Input node (Source) : It is a node that has only outgoing branches.

Output node (Sink) : It is a node that has only incoming branches.

Mixed node : It is a node that has both incoming and outgoing branches.

Path : A path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once.

Open path : A open path starts at a node and ends at another node.

Closed path : Closed path starts and ends at same node.

Forward path : It is a path from an input node to an output node that does not cross any node more than once.

***Forward path* gain** : It is the product of the branch transmittances (gains) of a forward path.

Individual loop : It is a closed path starting from a node and after passing through a certain part of a graph arrives at same node without crossing any node more than once.

Loop gain : It is the product of the branch transmittances (gains) of a loop.

Non-touching Loops : If the loops does not have a common node then they are said to be non- touching loops.

PROPERTIES OF SIGNAL FLOW GRAPH

The basic properties of signal flow graph are the following :

(i) The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect relationship.

(ii) Signal flow graph is applicable to linear systems only.

(iii) A node in the signal flow graph represents the variable or signal.

(iv) A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.

(v) A mixed node which has both incoming and outgoing signals can be treated as an output node by adding an outgoing branch of unity transmittance.

(vi) A branch indicates functional dependence of one signal on the other.

(vii) The signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

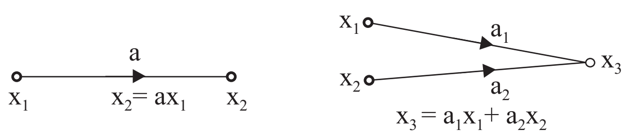
(viii) The signal flow graph of system is not unique. By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

SIGNAL FLOW GRAPH ALGEBRA

Signal flow graph for a system can be reduced to obtain the transfer function of the system using the following rules. The guideline in developing the rules for signal flow graph algebra is that the signal at a node is given by sum of all incoming signals.

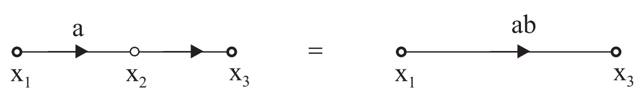
Rule 1 **:** Incoming signal to a node through a branch is given by the product of a signal at previous node and the gain of the branch.

Example:



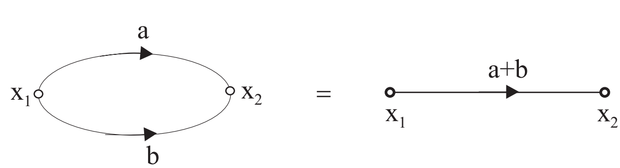
Rule 2 **:** Cascaded branches can be combined to give a single branch whose transmittance is equal to the product of individual branch transmittance.

Example:



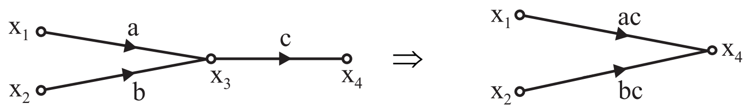
Rule 3 **:** Parallel branches may be represented by single branch whose transmittance is the sum of individual branch transmittances.

Example:



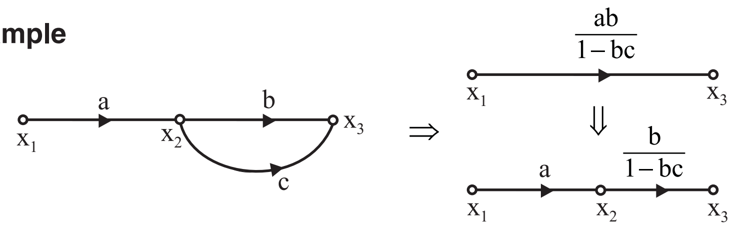
Rule 4 **:** A mixed node can be eliminated by multiplying the transmittance of outgoing branch (from the mixed node) to the transmittance of all incoming branches to the mixed node.

Example



Rule 5 **:** A loop may be eliminated by writing equations at the input and output node and rearranging the equations to find the ratio of output to input. This ratio gives the gain of resultant branch.

Example



Proof **:**

$x_2 = ax_1 + cx_3 \quad ; \quad x_3 = bx_2$

Put, $x_2 = ax_1 + cx_3$ in the equation for x_3.

$\therefore\ x_3 = b(ax_1 + cx_3) \quad \Rightarrow \quad x_3 = abx_1 + bcx_3 \quad \Rightarrow \quad x_3 - bcx_3 = abx_1 \quad \Rightarrow \quad x_3(1 - bc) = abx_1$

$$\therefore\ \frac{x_3}{x_1} = \frac{ab}{1 - bc}$$

SIGNAL FLOW GRAPH REDUCTION

The signal flow graph of a system can be reduced either by using the rules of a signal flow graph algebra or by using Mason's gain formula.

For signal flow graph reduction using the rules of signal flow graph, write equations at every node and then rearrange these equations to get the ratio of output and input (transfer function).

The signal flow graph reduction by above method will be time consuming and tedious. **S.J.Mason** has developed a simple procedure to determine the transfer function of the system represented as a signal flow graph. He has developed a formula called by his name **mason's gain formula** which can be directly used to find the transfer function of the system.

MASON'S GAIN FORMULA

The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let, $R(s)$ = Input to the system

$C(s)$ = Output of the system

Now, Transfer function of the system, $T(s) = \frac{C(s)}{R(s)}$ .....(1.34)

Mason's gain formula states the overall gain of the system [transfer function] as follows,

Overall gain, $T = \frac{1}{\Delta} \sum_k P_k \Delta_k$ .....(1.35) where, $T = T(s)$ = Transfer function of the system

P_K = Forward path gain of K^{th} forward path

K = Number of forward paths in the signal flow graph

$$\Delta = 1 - (\text{Sum of individual loop gains}) + \begin{pmatrix}\text{Sum of gain products of all possible}\\ \text{combinations of two non-touching loops}\end{pmatrix} - \begin{pmatrix}\text{Sum of gain products of all possible}\\ \text{combinations of three non-touching loops}\end{pmatrix} + \ldots\ldots$$

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path

CONSTRUCTING SIGNAL FLOW GRAPH FOR CONTROL SYSTEMS

A control system can be represented diagrammatically by signal flow graph. The differential equations governing the system are used to construct the signal flow graph.The following procedure can be used to construct the signal flow graph of a system.

1. Take Laplace transform of the differential equations governing the system in order to convert them to algebraic equations in s-domain.
2. The constants and variables of the s-domain equations are identified.
3. From the working knowledge of the system, the variables are identified as input, output and intermediate variables.
4. For each variable a node is assigned in signal flow graph and constants are assigned as the gain or transmittance of the branches connecting the nodes.
5. For each equation a signal flow graph is drawn and then they are interconnected to give overall signal flow graph of the system.

PROCEDURE FOR CONVERTING BLOCK DIAGRAM TO SIGNAL FLOW GRAPH

The signal flow graph and block diagram of a system provides the same information but there is no standard procedure for reducing the block diagram to find the transfer function of the system. Also the block diagram reduction technique will be tedious and it is difficult to choose the rule to be applied for simplification. Hence it will be easier if the block diagram is converted to signal flow graph and **Mason's gain formula** is applied to find the transfer function. The following procedure can be used to convert block diagram to signal flow graph.

1. Assume nodes at input, output, at every summing point, at every branch point and in between cascaded blocks.
2. Draw the nodes separately as small circles and number the circles in the order 1, 2, 3, 4, ... etc.
3. From the block diagram find the gain between each node in the main forward path and connect all the corresponding circles by straight line and mark the gain between the nodes.
4. Draw the feed forward paths between various nodes and mark the gain of feed forward path along with sign.
5. Draw the feedback paths between various nodes and mark the gain of feedback paths along with sign.

EXAMPLE 1.24

Construct a signal flow graph for armature controlled dc motor.

SOLUTION

The differential equations governing the armature controlled dc motor are (refer section 1.7).

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \quad ; \quad T = K_t i_a \quad ; \quad T = J\frac{d\omega}{dt} + B\omega \quad ; \quad e_b = K_b \omega \quad ; \quad \omega = d\theta/dt$$

On taking Laplace transform of above equations we get,

$$V_a(s) = I_a(s)\, R_a + L_a\, s\, I_a(s) + E_b(s) \qquad .....(1)$$

$$T(s) = K_t I_a(s) \qquad .....(2)$$

$$T(s) = J\, s\, \omega(s) + B\, \omega(s) \qquad .....(3)$$

$$E_b(s) = K_b\, \omega(s) \qquad .....(4)$$

$$\omega(s) = s\, \theta(s) \qquad .....(5)$$

The input and output variables of armature controlled dc motor are armature voltage $V_a(s)$ and angular displacement $\theta(s)$ respectively. The variables $I_a(s)$, $T(s)$, $E_b(s)$ and $\omega(s)$ are intermediate variables.

The equations (1) to (5) are rearranged & individual signal flow graph are shown in fig 1 to fig 5.

$$V_a(s) - E_b(s) = I_a(s)\,[R_a + s\, L_a]$$

$$\therefore\ I_a(s) = \frac{1}{R_a + sL_a}\,[\,V_a(s) - E_b(s)\,]$$

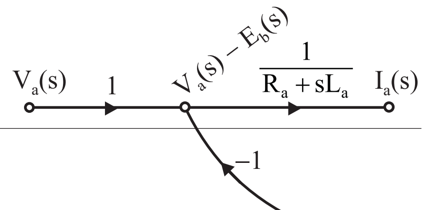


Fig 1

$$T(s) = K_t I_a(s)$$

Fig 2

$$T(s) = \omega(s)\,[Js + B]$$

$$\therefore\ \omega(s) = \frac{1}{Js + B}\, T(s)$$

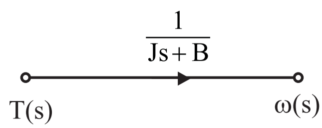


Fig 3

$$E_b(s) = K_b\, \omega(s)$$

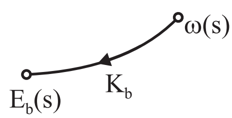


Fig 4

$$\omega(s) = s\theta(s)$$

$$\therefore\ \theta(s) = \frac{1}{s}\,\omega(s)$$

Fig 5

The overall signal flow graph of armature controlled dc motor is obtained by interconnecting the individual signal flow graphs shown in fig 1 to fig 5. The overall signal flow graph is shown in fig 6.

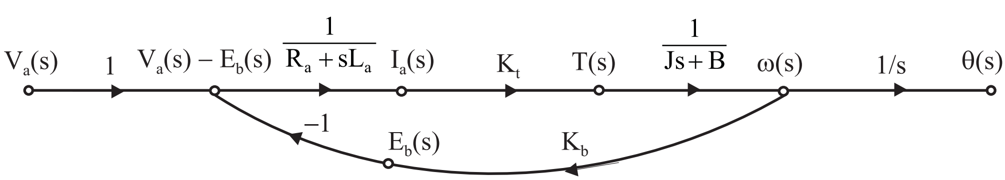
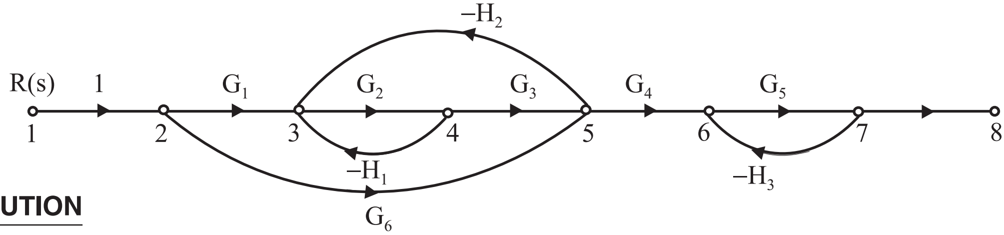


***Fig 6 :** Signal flow graph of armature controlled dc motor.*

EXAMPLE 1.25

Find the overall transfer function of the system whose signal flow graph is shown in fig 1.



SOLUTION

I. Forward Path Gains

There are two forward paths. $\therefore$ K = 2

Let forward path gains be P_1 and P_2.

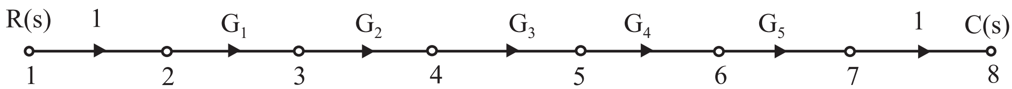


***Fig 2 :** Forward path-1.*

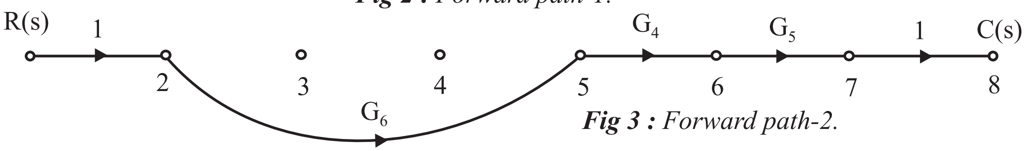


***Fig 3 :** Forward path-2.*

Gain of forward path-1, $P_1 = G_1\ G_2\ G_3\ G_4\ G_5$

Gain of forward path-2, $P_2 = G_4 G_5 G_6$

II. Individual Loop Gain

There are three individual loops. Let individual loop gains be P_{11}, P_{21} and P_{31}.

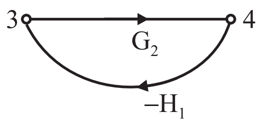


***Fig 4 :** Loop-1.*

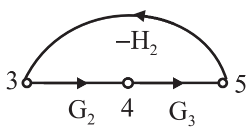


***Fig 5 :** Loop-2.*

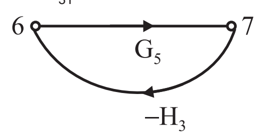


***Fig 6 :** Loop-3.*

Loop gain of individual loop-1, $P_{11} = -\ G_2 H_1$

Loop gain of individual loop-2, $P_{21} = -\ G_2 G_3 H_2$

Loop gain of individual loop-3, $P_{31} = -\ G_5 H_3$

III. Gain Products of Two Non-touching Loops

There are two combinations of two non-touching loops. Let the gain products of two non touching loops be P_{12} and P_{22}.

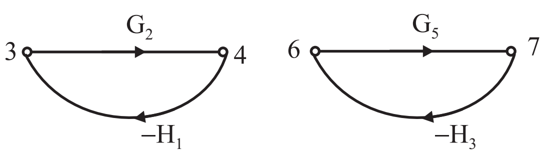


***Fig 7 :** First combination of 2 non-touching loops.*

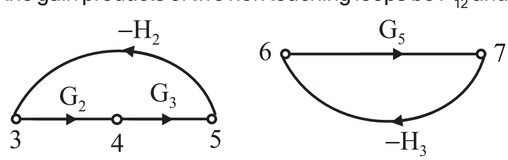


***Fig 8 :** Second combination of 2 non-touching loops.*

$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non touching loops}\end{array}\right\} P_{12} = P_{11} P_{31} = (-\ G_2 H_1)(-\ G_5 H_3) = G_2 G_5 H_1 H_3$$

$$\left.\begin{array}{l}\text{Gain product of second combination}\\ \text{of two non touching loops}\end{array}\right\} P_{22} = P_{21} P_{31} = (-\ G_2 G_3 H_2)(-\ G_5 H_3) = G_2 G_3 G_5 H_2 H_3$$

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$$
$$= 1 - (-G_2H_1 - G_2G_3H_2 - G_5H_3) + (G_2G_5H_1H_3 + G_2G_3G_5H_2H_3)$$
$$= 1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3$$

$\Delta_1 = 1$, Since there is no part of graph which is not touching with first forward path.

The part of the graph which is non touching with second forward path is shown in fig 9.

$$\Delta_2 = 1 - P_{11} = 1 - (-G_2H_1) = 1 + G_2H_1$$

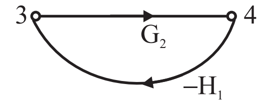


Fig 9

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad \text{(Number of forward paths is 2 and so K = 2)}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6(1 + G_2H_1)}{1 + G_2H_1 + G_2G_3H_2 + G_3H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6 + G_2G_4G_5G_6H_1}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

$$= \frac{G_2G_4G_5[G_1G_3 + G_6/G_2 + G_6H_1]}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

EXAMPLE 1.26

Find the overall gain of the system whose signal flow graph is shown in fig 1.

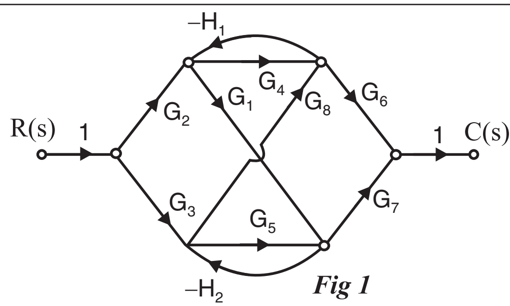


Fig 1

SOLUTION

Let us number the nodes as shown in fig 2.

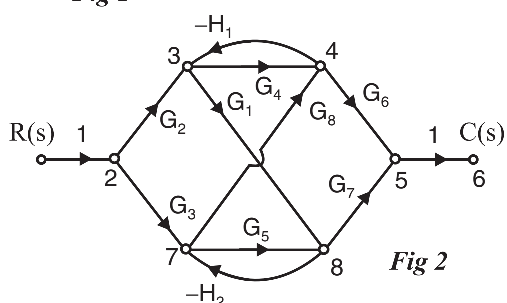


Fig 2

I. Forward Path Gains

There are six forward paths. $\therefore$ K = 6

Let the forward path gains be P^1, P^2, P^3, P^4, P^5 and P^6.

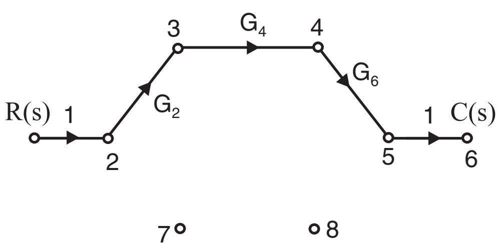


Fig 3 : *Forward path-1.*

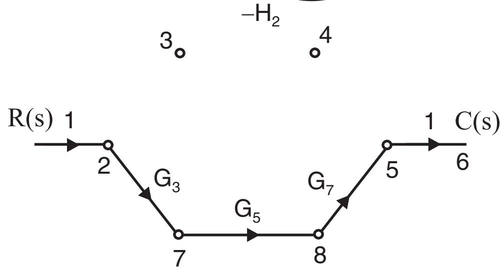


Fig 4 : *Forward path-2.*

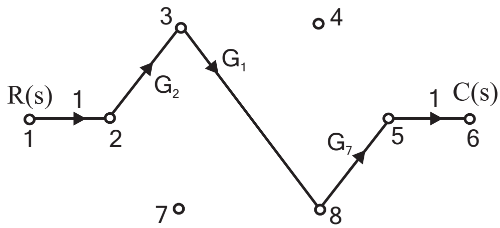


Fig 5 : Forward path-3

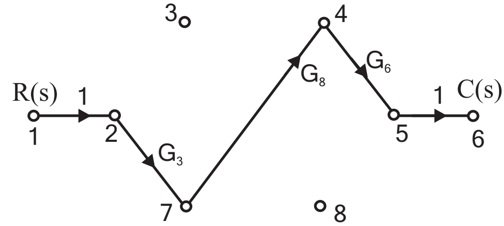


Fig 6 : Forward path-4

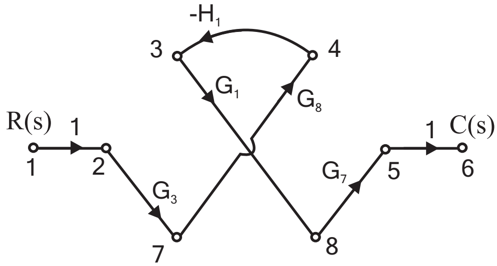


Fig 7 : Forward path-5

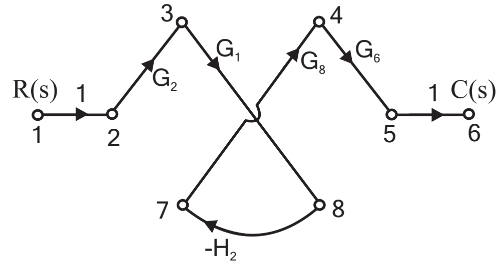


Fig 8 : Forward path-6

Gain of forward path-1, $P_1 = G_2G_4G_6$

Gain of forward path-2, $P_2 = G_3G_5G_7$

Gain of forward path-3, $P_3 = G_1G_2G_7$

Gain of forward path-4, $P_4 = G_3G_8G_6$

Gain of forward path-5, $P_5 = -G_1G_3G_7G_8H_1$

Gain of forward path-6, $P_6 = -G_1G_2G_6G_8H_2$

II. Individual Loop Gain

There are three individual loops.

Let individual loop gains be P_{11}, P_{21} and P_{31}.

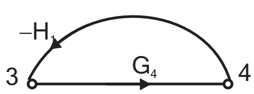


Fig 9 : Loop-1

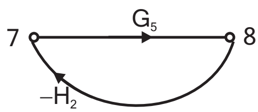


Fig 10 : Loop-2

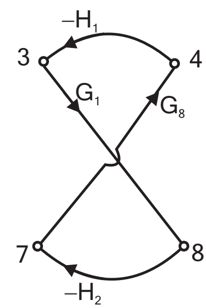


Fig 11 : Loop-3

Loop gain of individual loop-1, $P_{11} = -G_4H_1$

Loop gain of individual loop-2, $P_{21} = -G_5H_2$

Loop gain of individual loop-3, $P_{31} = G_1G_8H_1H_2$

III. Gain Products of Two Non-touching Loops

There is only one combination of two non-touching loops. Let gain product of two non-touching loops be P^{12}.

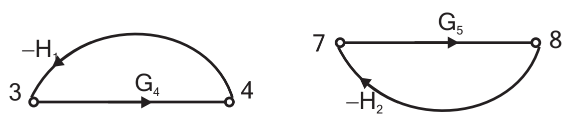


Fig 12 : Combination of 2 non-touching loops

$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non - touching loops}\end{array}\right\} P_{12} = P_{11}P_{21} = (-G_4H_1)(-G_5H_2) = G_4G_5H_1H_2$$

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = 1 - (-G_4H_1 - G_5H_2 + G_1G_8H_1H_2) + G_4G_5H_1H_2$$

$$= 1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2$$

The part of the graph non-touching forward path - 1 is shown in fig 13.

$\therefore \Delta_1 = 1 - (-G_5H_2) = 1 + G_5H_2$

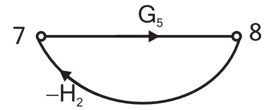


Fig 13

The part of the graph non-touching forward path -2 is shown in fig 14.

$\therefore \Delta_2 = 1 - (-G_4H_1) = 1 + G_4H_1$

There is no part of the graph which is non-touching with forward paths 3, 4, 5 and 6.

$\therefore \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$

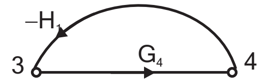


Fig 14

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\left(\sum_K P_K\Delta_K\right) \qquad \text{(Number of forward paths is six and so K = 6)}$$

$$= \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6)$$

$$= \frac{G_2G_4G_6(1+G_5H_2) + G_3G_5G_7(1+G_4H_1) + G_1G_2G_7 + G_3G_6G_8 - G_1G_3G_7G_8H_1 - G_1G_2G_6G_8H_2}{1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2}$$

EXAMPLE 1.27

Find the overall gain C(s)/R(s) for the signal flow graph shown in fig 1.

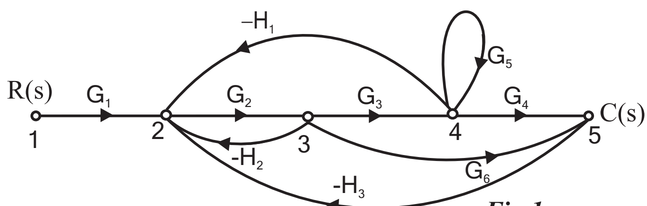


Fig 1

SOLUTION

I. Forward Path Gains

There are two forward paths. $\therefore$ K = 2. Let the forward path gains be P_1 and P_2.

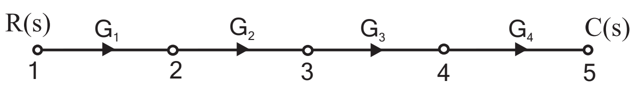


Fig 2 : *Forward path-1*

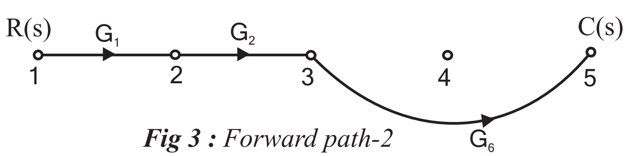


Fig 3 : *Forward path-2*

Gain of forward path-1, $P_1 = G_1G_2G_3G_4$

Gain of forward path-2, $P_2 = G_1G_2G_6$

II. Individual Loop Gain

There are five individual loops. Let the individual loop gains be p_{11}, p_{21}, p_{31}, p_{41} and p_{51}.

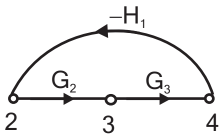


Fig 4 : loop-1

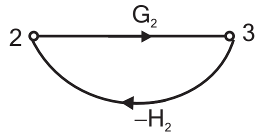


Fig 5 : loop-2

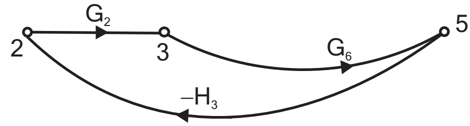


Fig 6 : loop-3

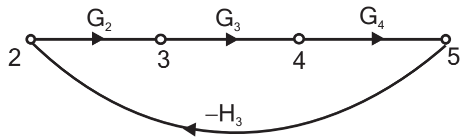


Fig 7 : loop-4

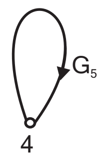


Fig 8 : loop-5

Loop gain of individual loop-1, $P_{11} = -G_2G_3H_1$

Loop gain of individual loop-2, $P_{21} = -H_2G_2$

Loop gain of individual loop-3, $P_{31} = -G_2G_6H_3$

Loop gain of individual loop-4, $P_{41} = -G_2G_3G_4H_3$

Loop gain of individual loop-5, $P_{51} = G_5$

III. Gain Products of Two Non-touching Loops

There are two combinations of two non-touching loops.

Let the gain products of two non-touching loops be P_{12} and P_{22}.

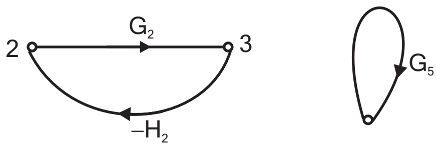


Fig 9 : First combination of two non-touching loops

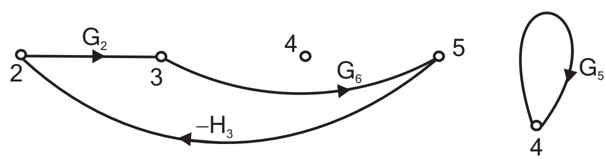


Fig 10 : Second combination of two non-touching loops

Gain product of first combination of two non touching loops $\Big\}$ $P_{12} = P_{21}P_{51} = (-G_2H_2)(G_5) = -G_2G_5H_2$

Gain product of second combination of two non touching loops $\Big\}$ $P_{22} = P_{31}P_{51} = (-G_2G_6H_3)(G_5) = -G_2G_5G_6H_3$

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51}) + (P_{12} + P_{22})$$

$$= 1 - (-G_2G_3H_1 - H_2G_2 - G_2G_3G_4H_3 + G_5 - G_2G_6H_3) + (-G_2H_2G_5 - G_2G_5G_6H_3)$$

Since there is no part of graph which is not touching forward path-1, $\Delta_1 = 1$.

The part of graph which is not touching forward path-2 is shown in fig 11.

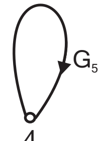


Fig 11

$$\therefore \Delta_2 = 1 - G_5$$

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K \quad \text{(Number of forward path is 2 and so K = 2)}$$

$$= \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2] = \frac{1}{\Delta}[G_1G_2G_3G_4 \times 1 + G_1G_2G_6(1 - G_5)]$$

$$= \frac{G_1G_2G_3G_4 + G_1G_2G_6 - G_1G_2G_5G_6}{1 + G_2G_3H_1 + H_2G_2 + G_2G_3G_4H_3 - G_5 + G_2G_6H_5 - G_2G_6H_3 - G_2H_2G_5 - G_2G_5G_6H_3}$$

EXAMPLE 1.28

Find the overall gain C(s)/R(s) for the signal flow graph shown in fig 1.

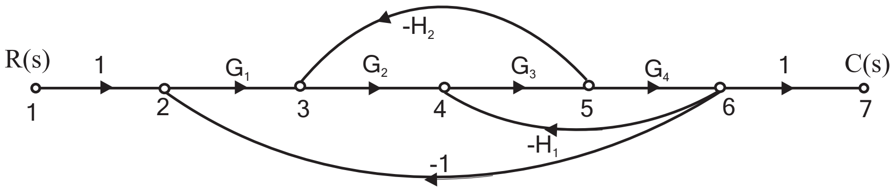


Fig 1

SOLUTION

I. Forward Path Gains

There is only one forward path. $\therefore$ K = 1.

Let the forward path gain be P_1.

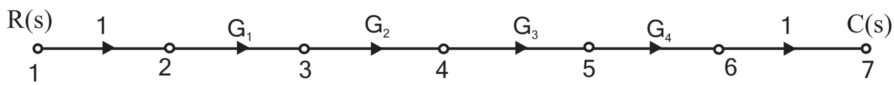


Fig 2 : *Forward path-1*

Gain of forward path-1, $P_1 = G_1G_2G_3G_4$

II. Individual Loop Gain

There are three individual loops. Let the loop gains be P_{11}, P_{21}, P_{31}.

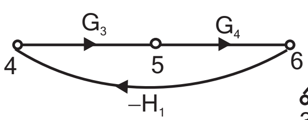


Fig 3 : *loop-1*

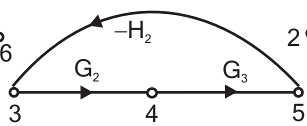


Fig 4 : *loop-2*

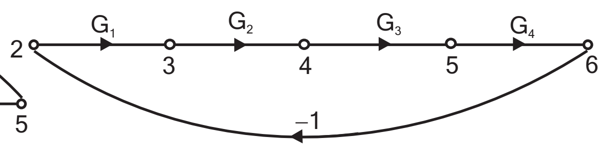


Fig 5 : *loop-3*

Loop gain of individual loop-1, $P_{11} = -G_3G_4H_1$

Loop gain of individual loop-2, $P_{21} = -G_2G_3H_2$

Loop gain of individual loop-3, $P_{31} = -G_1G_2G_3G_4$

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31})$$

$$= 1 - (-G_3G_4H_1 - G_2G_3H_2 - G_1G_2G_3G_4)$$

$$= 1 + G_3G_4H_1 + G_2G_3H_2 + G_1G_2G_3G_4$$

Since no part of the graph is non-touching with forward path-1, $\Delta_1 = 1$.

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_K P_K \Delta_K = \frac{1}{\Delta} P_1 \Delta_1 \text{ (Number of forward path is 1 and so K = 1)}$$

$$= \frac{G_1G_2G_3G_4}{1 + G_3G_4H_1 + G_2G_3H_2 + G_1G_2G_3G_4}$$

EXAMPLE 1.29

The signal flow graph for a feedback control system is shown in fig 1. Determine the closed loop transfer function C(s)/R(s).

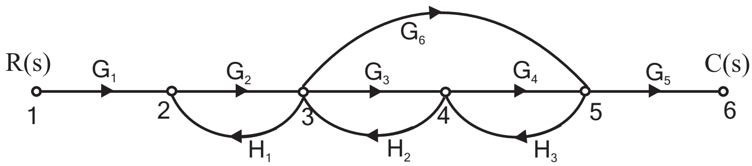


Fig 1

SOLUTION

I. Forward Path Gains

There are two forward paths. $\therefore$ K = 2.

Let forward path gains be P_1 and P_2.

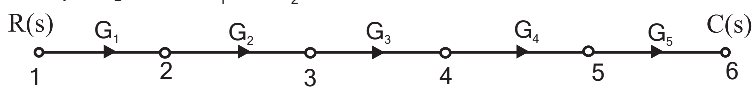


***Fig 2 :** Forward path-1*

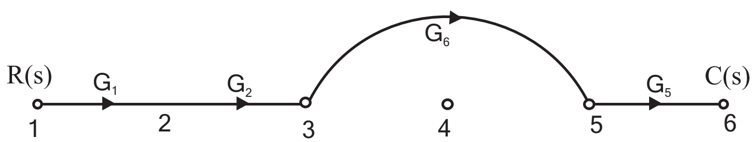


***Fig 3 :** Forward path-2*

Gain of forward path-1, $P_1 = G_1G_2G_3G_4G_5$

Gain of forward path-2, $P_2 = G_1G_2G_6G_5$

II. Individual Loop Gain

There are four individual loops. Let individual loop gains be P_{11}, P_{21}, P_{31} and P_{41}.

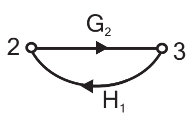


***Fig 4 :** loop-1*

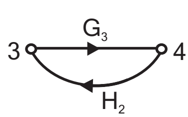


***Fig 5 :** loop-2*

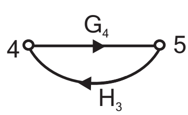


***Fig 6 :** loop-3*

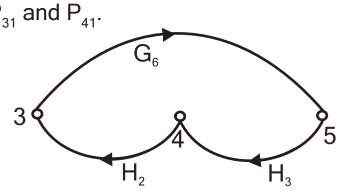


***Fig 7 :** loop-4*

Loop gain of individual loop-1, $P_{11} = G_2H_1$

Loop gain of individual loop-2, $P_{21} = G_3H_2$

Loop gain of individual loop-3, $P_{31} = G_4H_3$

Loop gain of individual loop-4, $P_{41} = G_6H_2H_3$

III. Gain Products of Two Non-touching Loops

There is only one combination of two non-touching loops. Let the gain products of two non-touching loops be P_{12}.

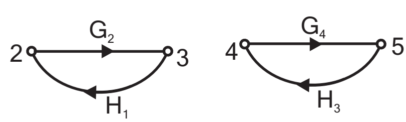


***Fig 8 :** First combination of two non touching loops*

Gain product of first combination of two non - touching loops $\Big\}\ P_{12} = (G_2H_1)(G_4H_3)$

$$= G_2G_4H_1H_3$$

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41}) + P_{12}$$

$$= 1 - (G_2H_1 + G_3H_2 + G_4H_3 + G_6H_2H_3) + G_2G_4H_1H_3$$

$$= 1 - G_2H_1 - G_3H_2 - G_4H_3 - G_6H_2H_3 + G_2G_4H_1H_3$$

Since there is no part of graph which is non-touching with forward path-1 and 2, $\Delta_1 = \Delta_2 = 1$

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad \text{(Number of forward paths is two and so K = 2)}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_1G_2G_5G_6}{1 - G_2H_1 - G_3H_2 - G_4H_3 - G_6H_2H_3 + G_2G_4H_1H_3}$$

EXAMPLE 1.30

Convert the given block diagram to signal flow graph and determine C(s)/R(s).

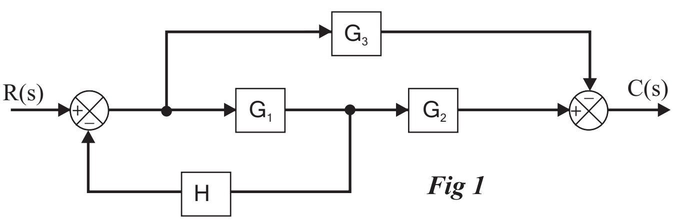


Fig 1

SOLUTION

The nodes are assigned at input, output, at every summing point & branch point as shown in fig 2.

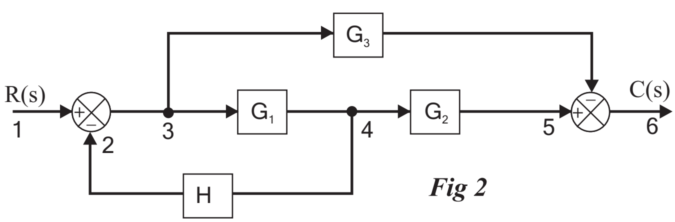


Fig 2

The signal flow graph of the above system is shown in fig 3.

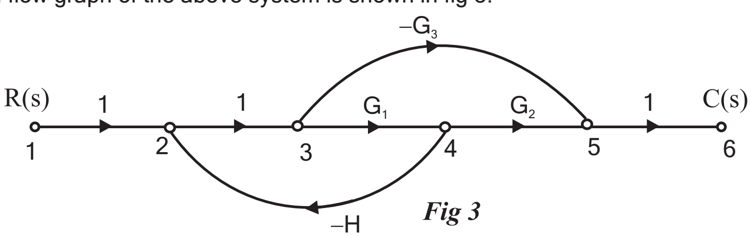


Fig 3

I. Forward Path Gains

There are two forward paths. $\therefore$ K = 2

Let the forward path gains be P_1 and P_2.

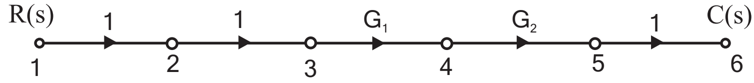


***Fig 4 :** Forward path-1*

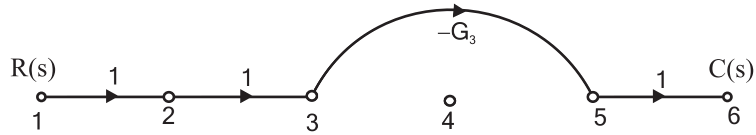


***Fig 5 :** Forward path-2*

Gain of forward path-1, $P_1 = G_1G_2$

Gain of forward path-2, $P_2 = -G_3$

II. Individual Loop Gain

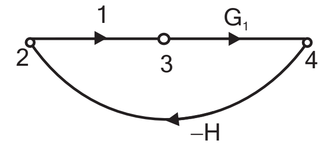


***Fig 6 :** loop-1*

There is only one individual loop. Let the individual loop gain be P_{11}.

Loop gain of individual loop-I, $P_{11} = -G_1H$.

III. Gain Products of Two Non-touching Loops

There are no combinations of non-touching loops.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - [P_{11}] = 1 + G_1H$$

Since there are no part of the graph which is non-touching with forward path-1 and 2,

$$\Delta_1 = \Delta_2 = 1$$

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2] = \frac{G_1G_2 - G_3}{1 + G_1H}$$

EXAMPLE 1.31

Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula.

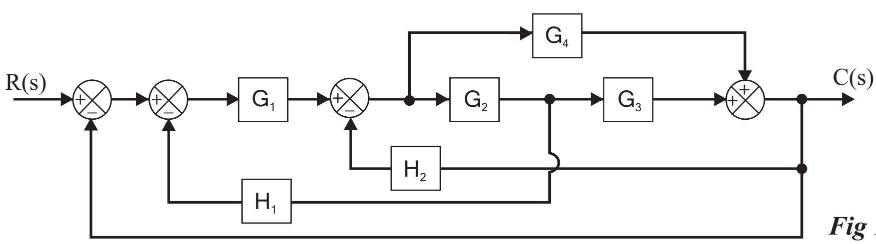


Fig 1

SOLUTION

The nodes are assigned at input, ouput, at every summing point & branch point as shown in fig 2.

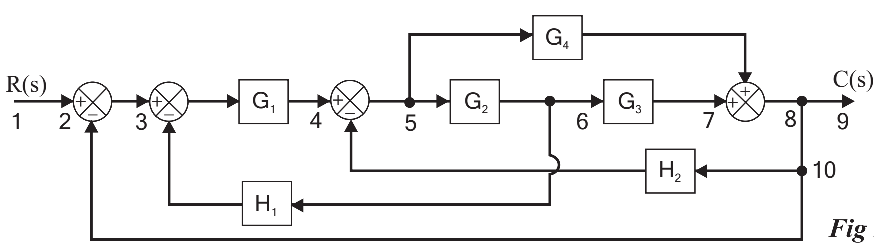


Fig 2

The signal flow graph for the above block diagram is shown in fig 3.

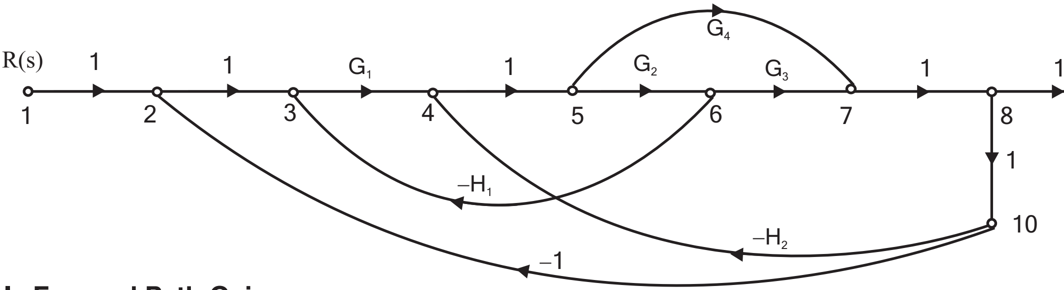


Fig 3

I. Forward Path Gains

There are two forward paths. $\therefore$ K=2.

Let the gain of the forward paths be P_1 and P_2.

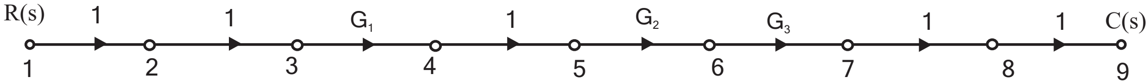


***Fig 4** : Forward path-1*

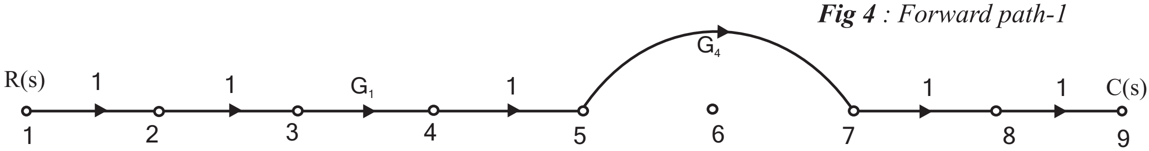


***Fig 5** : Forward path-2*

Gain of forward path-1, $P_1 = G_1G_2G_3$

Gain of forward path-2, $P_2 = G_1G_4$

II. Individual Loop Gain

There are five individual loops. Let the individual loop gain be P_{11}, P_{21}, P_{31}, P_{41} and P_{51}.

Loop gain of individual loop-1, $P_{11} = -\,G_1G_2G_3$

Loop gain of individual loop-2, $P_{21} = -\,G_2G_1H_1$

Loop gain of individual loop-3, $P_{31} = -\,G_2G_3H_2$

Loop gain of individual loop-4, $P_{41} = -\,G_1G_4$

Loop gain of individual loop-5, $P_{51} = -\,G_4H_2$

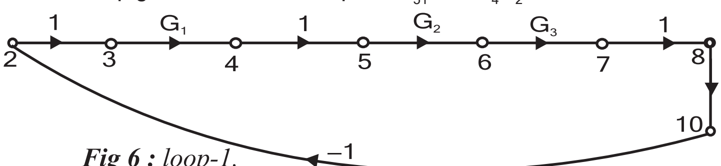


***Fig 6 :** loop-1.*

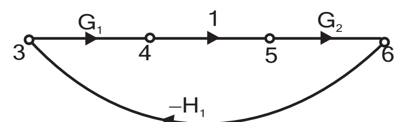


***Fig 7 :** loop-2.*

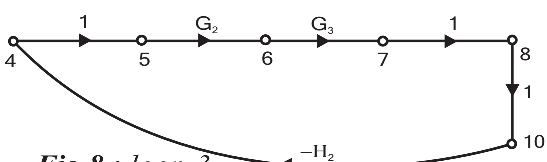


***Fig 8 :** loop-3.*

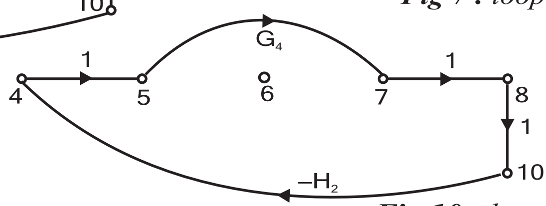


***Fig 10 :** loop-5.*

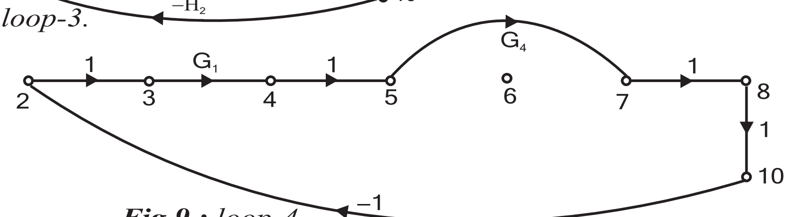


***Fig 9 :** loop-4.*

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.,.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - [\,P_{11} + P_{21} + P_{31} + P_{41} + P_{51}] = 1 + G_1G_2G_3 + G_1G_2H_1 + G_2G_3H_2 + G_1G_4 + G_4H_2$$

Since no part of graph is non touching with forward paths-1 and 2, $\Delta_1 = \Delta_2 = 1$.

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2]$$

$$= \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 + G_1G_2H_1 + G_2G_3H_2 + G_1G_4 + G_4H_2}$$

EXAMPLE 1.32

Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula.

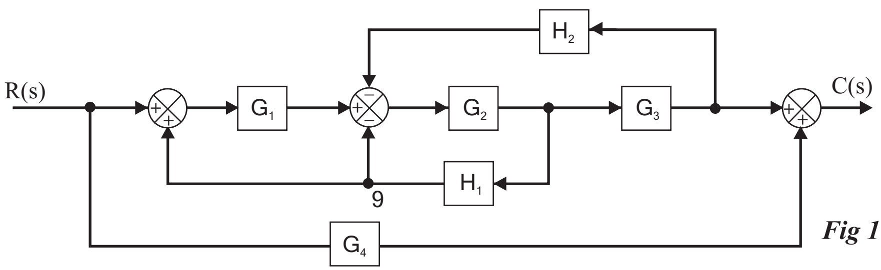


Fig 1

SOLUTION

The nodes are assigned at input, output, at every summing point & branch point as shown in fig 2.

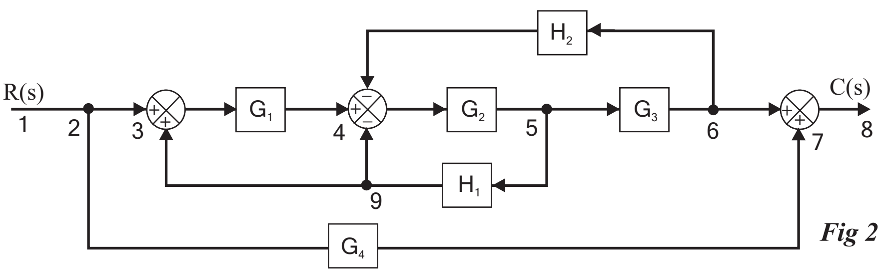


Fig 2

The signal flow graph for the above block diagram is shown in fig 3.

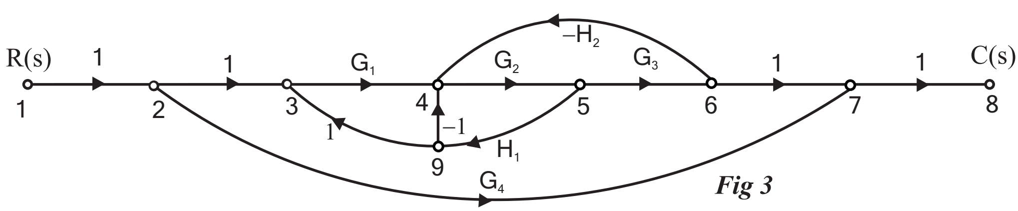


Fig 3

I. Forward Path Gains

There are two forward path, $\therefore$ K=2.

Let the forward path gains be P_1 and P_2.

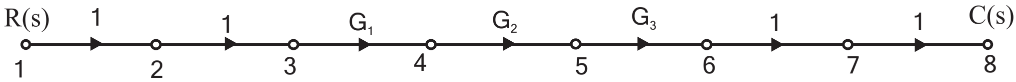


Fig 4 : *Forward path-1.*

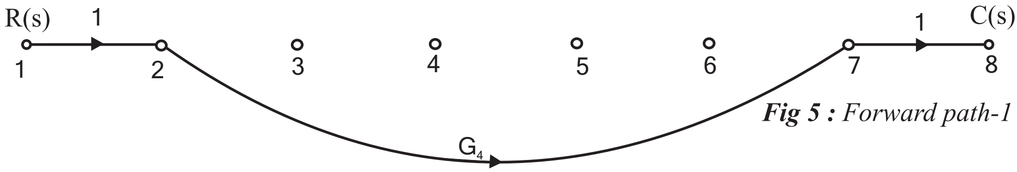


Fig 5 : *Forward path-1*

Gain of forward path-1, $P_1 = G_1G_2G_3$

Gain of forward path-2, $P_2 = G_4$

II. Individual Loop Gain

There are three individual loops with gains P_{11}, P_{21} and P_{31}.

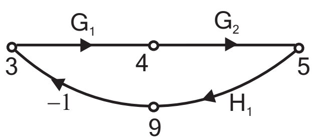


Fig 6 : *loop-1.*

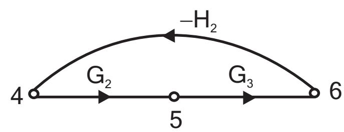


Fig 7 : *loop-2.*

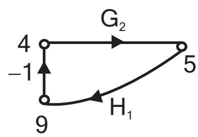


Fig 8 : *loop-3.*

Gain of individual loop-1, $P_{11} = -G_1G_2H_1$

Gain of individual loop-2, $P_{21} = -G_2G_3H_2$

Gain of individual loop-3, $P_{31} = -G_2H_1$

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two-non touching loops, three non-touching loops, etc.,.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - [P_{11} + P_{21} + P_{31}] = 1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1$$

Since no part of graph touches forward path-1, $\Delta_1 = 1$.

The part of graph non touching forward path-2 is shown in fig 9.

$$\therefore \Delta_2 = 1 - [-G_1G_2H_1 - G_2G_3H_2 - G_2H_1]$$

$$= 1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1$$

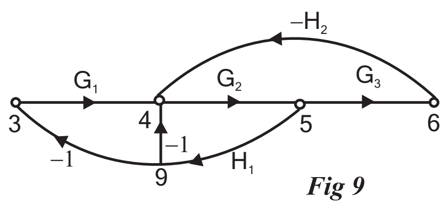


Fig 9

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2] \quad \text{(Number of forward paths is 2 and so K = 2)}$$

$$= \frac{1}{\Delta}[G_1G_2G_3 + G_4(1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1)]$$

$$= \frac{1}{\Delta}[G_1G_2G_3 + G_4 + G_1G_2H_1 + G_2G_3G_4H_2 + G_2G_4H_1]$$

$$= \frac{G_1G_2G_3 + G_4 + G_1G_2G_4H_1 + G_2G_3G_4H_2 + G_2G_4H_1}{1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1}$$

EXAMPLE 1.33

Draw a signal flow graph and evaluate the closed loop transfer function of a system whose block diagram is shown in fig 1.

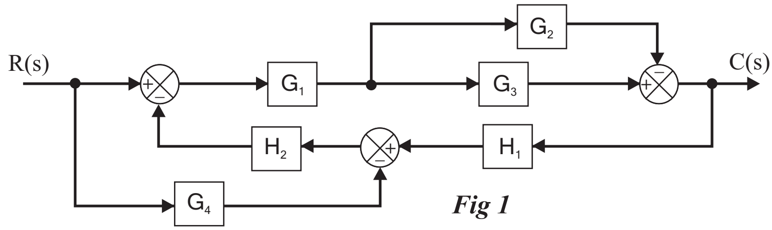


Fig 1

SOLUTION

The nodes are assigned at input, output, at every summing point & branch point as shown in fig 2.

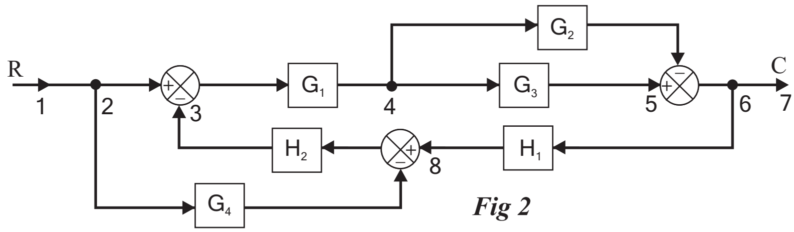


Fig 2

The signal flow graph for the block diagram of fig 2, is shown in fig 3.

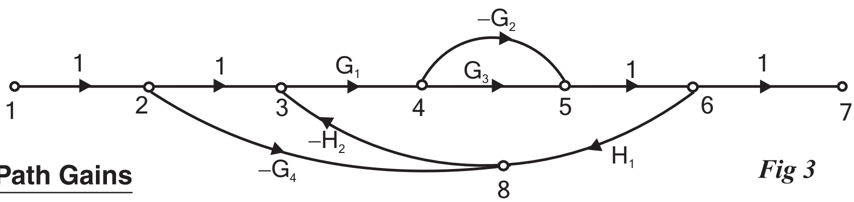


Fig 3

I. Forward Path Gains

There are four forward paths, $\therefore$ K = 4

Let the forward path gains be P_1, P_2, P_3 and P_4.

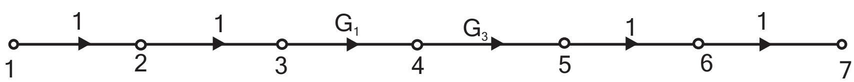


***Fig 4 :** Forward path-1.*

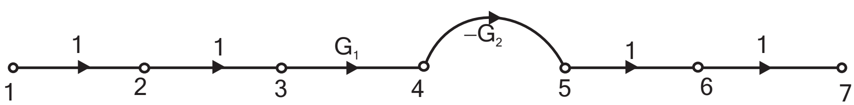


***Fig 5 :** Forward path-2.*

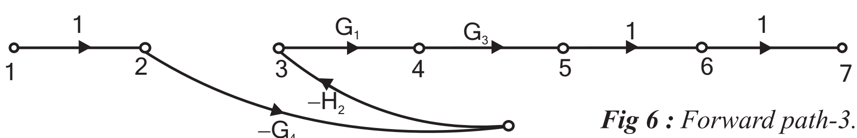


***Fig 6 :** Forward path-3.*

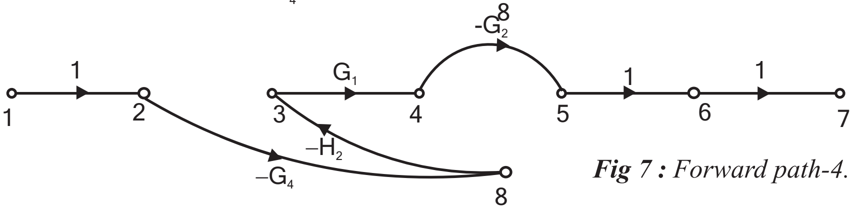


***Fig 7 :** Forward path-4.*

Gain of forward path-1, $P_1 = G_1G_3$
Gain of forward path-2, $P_2 = -G_1G_2$
Gain of forward path-3, $P_3 = G_1G_3G_4H_2$
Gain of forward path-4, $P_4 = -G_1G_2G_4H_2$

II. Individual Loop Gain

There are two individual loops, let individual loop gains be P_{11} and P_{21}.

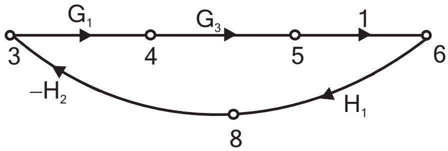


***Fig 8 :** loop-1*

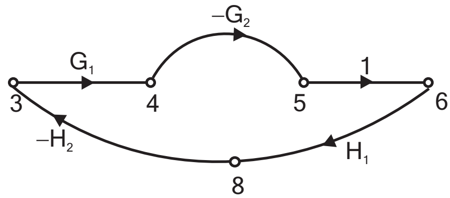


***Fig 9 :** loop-2*

Loop gain of individual loop-1, $P_{11} = -G_1G_3H_1H_2$

Loop gain of individual loop-2, $P_{21} = G_1G_2H_1H_2$

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.,.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - [\text{ sum of individual loop gain }] = 1 - (P_{11} + P_{21})$$
$$= 1 - [-G_1G_3H_1H_2 + G_1G_2H_1H_2] = 1 + G_1G_3H_1H_2 - G_1G_2H_1H_2$$

Since no part of graph is non touching with the forward paths, $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$.

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{P_1 + P_2 + P_3 + P_4}{\Delta} \quad \text{(Number of forward paths is 4 and so K = 4)}$$

$$= \frac{G_1G_3 - G_1G_2 + G_1G_3G_4H_2 - G_1G_2G_4H_2}{1 + G_1G_3H_1H_2 - G_1G_2H_1H_2}$$

$$= \frac{G_1(G_3 - G_2) + G_1G_4H_2(G_3 - G_2)}{1 + G_1H_1H_2(G_3 - G_2)} = \frac{G_1(G_3 - G_2)(1 + G_4H_2)}{1 + G_1H_1H_2(G_3 - G_2)}$$

1.13 THERMAL SYSTEM

List of symbols used in thermal system

q	=	Heat flow rate, Kcal/sec
θ_1	=	Absolute temperature of emitter, °K
θ_2	=	Absolute temperature of receiver, °K
$\Delta\theta$	=	Temperature difference, ºC
A	=	Area normal to heat flow, m^2
K	=	Conduction or Convection coefficient, Kcal/sec-ºC
K_r	=	Radiation coefficient, Kcal/sec-ºC
H	=	K/A=Convection coefficient, Kcal/m^2-sec-ºC
K	=	Thermal conductivity , Kcal/m-sec-ºC
ΔX	=	Thickness of conductor, m
R	=	Thermal resistance, ºC-sec/Kcal
C	=	Thermal capacitance, Kcal/ºC

HEAT FLOW RATE

Thermal systems are those that involve the transfer of heat from one substance to another. There are three different ways of heat flow from one substance to another. They are conduction, convection and radiation.

For conduction,

$$\text{Heat flow rate, } q = K \triangle \theta = \frac{KA}{\triangle X} \quad .....(1.36)$$

For convection,

$$\text{Heat flow rate, } q = K \Delta\theta = HA \Delta\theta \quad .....(1.37)$$

For radiation,

$$\text{Heat flow rate, } q = K_r (\theta_1^4 - \theta_2^4)$$

$$\text{If } \theta_1 >> \theta_2 \text{ then } q = K_f \bar{\theta}^4 \quad .....(1.38)$$

$$\text{Where } \bar{\theta}^4 = (\theta_1^4 - \theta_2^4)^{\frac{1}{4}}$$

Note : *$\bar{\theta}_4$ is called effective temperature difference of the emitter and receiver.*

BASIC ELEMENTS OF THERMAL SYSTEM

The model of thermal systems are obtained by using thermal resistance and capacitance which are the basic elements of the thermal system.

The thermal resistance and capacitance are distributed in nature. But for simplicity in analysis lumped parameter models are used. In lumped parameter model it is assumed that the substances that are characterized by resistance to heat flow have negligible heat capacitance and the substances that are characterized by heat capacitance have negligible resistance to heat flow.

The ***thermal resistance***, R for heat transfer between two substances is defined as the ratio of change in temperature and change in heat flow rate.

$$\boxed{\text{Thermal resistance, } R = \frac{\text{Change in Temperature, } ^\circ C}{\text{Change in heat flow rate, Kcal/sec}}}$$

For conduction or convection,

$$\text{Heat flow rate, } q = K \Delta\theta$$

On differentiating we get,

$$dq = K\, d(\Delta\theta)$$

$$\therefore \frac{d(\triangle\theta)}{dq} = \frac{1}{K}$$

$$\text{But thermal resistance , } R = \frac{d(\triangle\theta)}{dq}$$

$$\therefore \text{ Thermal resistance, } R = \frac{1}{K} \text{ for conduction}$$

$$= \frac{1}{K} = \frac{1}{HA} \text{ for convection}$$

For radiation,

$$\text{Heat flow rate, } q = K_r \bar{\theta}^4$$

On differentiating we get

$$dq = K_r\, 4\, \bar{\theta}^3\, d\bar{\theta}$$

$$\therefore \frac{d\bar{\theta}}{dq} = \frac{1}{K_r\, 4\bar{\theta}^3}$$

$$\text{But thermal resistance, } R = \frac{d\bar{\theta}}{dq}$$

$$\therefore \text{ Thermal resistance (for radiation), } R = \frac{1}{4\, K_r\, \bar{\theta}^3}$$

Thermal capacitance, C is defined as the ratio of change in heat stored and change in temperature

$$\boxed{\text{Thermal capacitance, } C = \frac{\text{Change in heat stored, Kcal}}{\text{Change in temparature, } ^\circ C}} \quad .....(1.39)$$

Let M = Mass of substance considered, Kg

c_p = Specific heat of substance, Kcal/Kg -°C

Now, Thermal capacitance, $C = Mc_p$ .....(1.40)

EXAMPLE OF THERMAL SYSTEM

Consider a simple thermal system shown in fig 1.28. Let us assume that the tank is insulated to eliminate heat loss to the surrounding air, there is no heat storage in the insulation and liquid in the tank is kept at uniform temperature by perfect mixing with the help of a stirrer. Thus, a single temperature is used to describe the temperature of the liquid in the tank and of the outflowing liquid. The transfer function of this system can be derived as shown below.

Let $\bar{\theta}_i$ = Steady state temperature of inflowing liquid, °C

$\bar{\theta}_0$ = Steady state temperature of outflowing liquid, °C

G = Steady state liquid flow rate, Kg/sec

M = Mass of liquid in tank, Kg

c = Specific heat of liquid, Kcal/Kg °C

R = Thermal resistance, °C - sec/Kcal

C = Thermal capacitance, Kcal/°C

Q = Steady state heat input rate, Kcal/sec

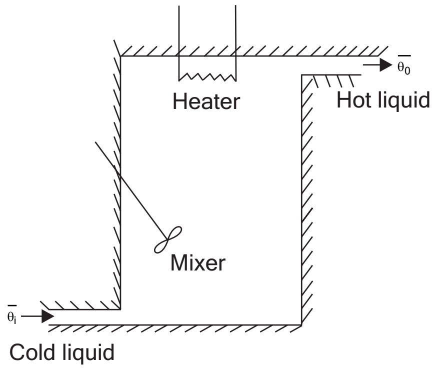


***Fig 1.28 :** Thermal system.*

Let us assume that the temperature of inflowing liquid is kept constant. Let the heat input rate to the system supplied by the heater is suddenly changed from $\bar{Q}$ to $\bar{Q} + q_0$. Due to this, the heat output flow rate will gradually change from $\bar{Q}$ to $\bar{Q} + q_i$ The temperature of the outflowing liquid will also be changed from $\bar{\theta}_0$ to $\bar{\theta}_0 + \theta$

For this system the equation for q_0, C and R are obtained as follows,

$$\left.\begin{array}{l}\text{Change in output}\\ \text{heat flow rate}\end{array}\right\} q_0 = \begin{array}{c}\text{Liquid flow}\\ \text{rare, G}\end{array} \times \begin{array}{c}\text{Specific heat of}\\ \text{liquid, c}\end{array} \times \begin{array}{c}\text{change in}\\ \text{temperature}, \theta\end{array}$$

$$= Gc\theta \qquad .....(1.41)$$

Thermal capacitance, C = Mass, M x Specific heat of liquid, c = Mc .....(1.42)

$$\boxed{\text{Thermal resis tan ce, } R = \frac{\text{Change in temperature}, \theta}{\text{Change in heat rate, } q_0} = \frac{\theta}{q_0}} \qquad .....(1.43)$$

On substituting for q_0 from equation (1.41) in equation (1.43) we get,

$$R = \frac{\theta}{Gc\theta} = \frac{1}{Gc} \qquad .....(1.44)$$

In this system, rate of change of temperature is directly proportional to change in heat input rate.

$$\therefore \frac{d\theta}{dt} \alpha \; q_i - q_0 \;\; ; \text{ the constant of proportionality is capacitance C of the system.}$$

$$\therefore C\frac{d\theta}{dt} = q_i - q_0 \qquad .....(1.45)$$

Equation (1.45) is the differential equation governing the system. Since equation (1.45) is of first order equation, the system is first order system.

From equation (1.43), $R = \frac{\theta}{q_0} \qquad \therefore q_0 = \frac{\theta}{R}$ .....(1.46)

On substituting for q_0 from equation (1.46) in equation (1.45) we get,

$$C\frac{d\theta}{dt} = q_i \quad \Rightarrow \quad C\frac{d\theta}{dt} = \frac{Rq_i - \theta}{R} \quad \Rightarrow \quad RC\frac{d\theta}{dt} = Rq_i - \theta$$

$$\therefore \ RC\frac{d\theta}{dt} + \theta = Rq_i \qquad ....(1.47)$$

Let, $\mathcal{L}\{\theta\} = \theta(s)$; $\quad \mathcal{L}\left\{\frac{d\theta}{dt}\right\} = s\,\theta(s)$; $\quad \mathcal{L}\{q_i\} = Q(s)$

On taking Laplace transform of equation (1.47) we get,

$$RC\, s\,\theta(s) + \theta(s) = R\, Q_i(s)$$

$$\theta(s)\,[sRC + 1] = R\, Q_i(s)$$

$\frac{\theta(s)}{Q_i(s)}$ is the required transfer function of the system

$$\therefore \ \frac{\theta(s)}{Q_i} = \frac{R}{sRC+1} = \frac{R}{RC\left(s + \frac{1}{RC}\right)} = \frac{\frac{1}{C}}{s + \frac{1}{RC}} \qquad .... (1.48)$$

1.14 HYDRAULIC SYSTEM

The Hydraulic system of interest to control engineers may be classified into,

1. Liquid Level system and
2. Hydraulic devices

The liquid level system consists of storage tanks and connecting pipes. The variables to be controlled are liquid height in tanks and flow rate in pipes. The driving force is the relative difference of the liquid heights in the tanks.

The Hydraulic devices are devices using incompressible oil as their working medium. These devices are used for controlling the forces and motions. The driving force is the high pressure oil supplied by the Hydraulic pumps.

Liquids are slightly compressible at high pressures. In hydraulic system, the compressibility effects may be neglected and conservation of volume is used as the basic physical law. The variables of hydraulic system are volumetric flow rate, q and pressure, P. The volumetric flow rate, q is through variable and it is analogous to current. The pressure, P is across variable and it is analogous to voltage.

Three basic elements of hydraulic systems are the ***Resistance, Capacitance and Inertance***. The liquid flowing out of a tank can meet the resistance in several ways. Liquid while flowing through a pipe meet with resistance due to the friction between pipe walls and liquid. Presence of valves, bends, coupling of pipe of different diameter also offer resistance to liquid flow.

The ***capacitance*** is an energy storage element and it represents storage in gravity field. The ***inertance*** represents fluid inertia and is derived from the inertia forces required to accelerate the fluid in a pipe. It is also an energy storage element. But the energy storage due to inertance element is negligible compared to that of capacitance element.

Consider the flow through a short pipe connecting two tanks. The ***Resistance***, R for liquid flow in such a pipe or restriction, is defined as the change in the level difference, necessary to cause a unit change in the flow rate.

$$\boxed{R = \frac{\text{Change in level differenc, m}}{\text{Change in flow rate, m}^3/\text{sec}}}$$

The ***Capacitance***, C of a tank is defined to be the change in quantity of stored liquid, necessary to cause a unit change in the potential (head).

$$\boxed{C = \frac{\text{Chnge in liquid stored}, m^3}{\text{Change in head}, m}}$$

EXAMPLE OF LIQUID LEVEL SYSTEM

A simple liquid level system is shown in figure 1.29 with steady state flow rate, $\bar{Q}$ and steady state head, $\bar{H}$.

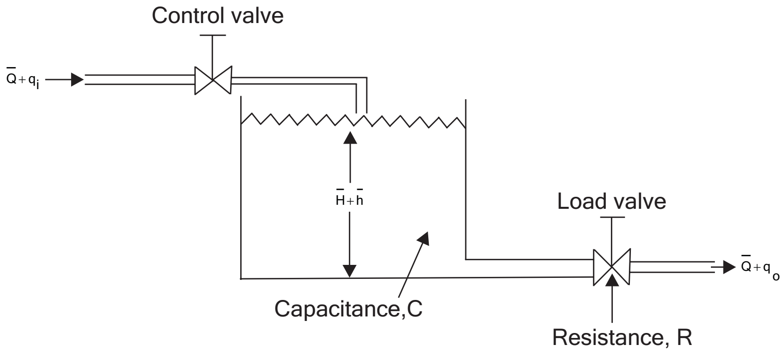


***Fig 1.29:**Liquid level system*

Let, $\bar{Q}$ = Steady-state flow rate (before any change has occured), m^3/sec

q_i = Small deviation of inflow rate from its steady-state value, m^3/sec

q_0 = Small deviation of outflow rate from its steady-state value, m^3/sec

$\bar{H}$ = Steady state head (before any change has occured), m

h = Small deviation of head from its steady-state value, m

Let the system be considered linear. The differential equation governing the system is obtained by equating, the change in flow rate to the amount stored in the tank. In a small time interval dt, let the change in flow rate be $(q_i – q_0)$, and the change in height be dh.

Now, Change in storage = Change in flow rate

$$\therefore C\,dh = (q_i – q_0)\,dh \qquad .....(1.49)$$

$$\text{The resistance, } R = \frac{\text{Change in head}}{\text{Change in outflow rate}} = \frac{h}{q_0}$$

$$\therefore q_0 = \frac{h}{R} \qquad .....(1.50)$$

On substituting for q_0 from equation (1.50) in equation (1.49) we get,

$$C\,dh = \left(q_i - \frac{h}{R}\right)dt \;\Rightarrow\; C\,dh = \left(\frac{q_i R - h}{R}\right)dt \;\Rightarrow\; RC\frac{dh}{dt} = q_i R - h$$

$$RC\frac{dh}{dt} + h = q_i R \qquad .....(1.51)$$

The equation (1.51) is the differential equation governing the system. The term RC is the time constant of the system. On taking Laplace's transform of equation (1.51), we get,

$$\boxed{\textbf{\textit{Note:}}\; \mathcal{L}\{h\} = H(s);\quad \mathcal{L}\left\{\frac{dh}{dt}\right\} = sH(s);\quad \mathcal{L}\{q_i\} = Q_i(s)}$$

$$RC\,sH(s) + H(s) = Q_i(s)R$$

$$(s\,RC + 1)\,H(s) = Q_i(s)\,R$$

$$\therefore \frac{H(s)}{Q_i(s)} = \frac{R}{(sRC+1)} = \frac{R}{RC(s+1/RC)} = \frac{1/C}{s+1/RC} \qquad .....(1.52)$$

The equation (1.52) is the required transfer function of the system.

HYDRAULIC DEVICES

The hydraulic devices are used in hydraulic feedback systems and in combined electro-mechanical-hydraulic systems. In hydraulic devices, power is transmitted through the action of fluid flow under pressure and the fluid is incompressible. The fluid used are petroleum based oils or non-inflammable synthetic oils.

The hydraulic devices used in control systems are generally classified as hydraulic motors and hydraulic linear actuators.

The output of hydraulic motor is rotary motion and that of linear actuator is translational. The hydraulic motor is physically smaller in size than an electric motor for the same power output. Also, the hydraulic components are more rugged than the corresponding electrical components. The applications of hydraulic devices are power steering and brakes in automobiles, the steering mechanism of large ships, the control of large machine tools, etc.

ADVANTAGES OF HYDRAULIC DEVICES

(i) Hydraulic fluid acts as a lubricant and coolant.

(ii) Comparatively small sized hydraulic actuators can develop large forces or torques.

(iii) Hydraulic actuators can be operated under continuous, intermittent, reversing and stalled conditions without damage.

(iv) Hydraulic actuators have a higher speed of response. They offer fast starts, stops and speed reversals.

(v) With availability of both linear and rotary actuators, the design has become more flexible.

(vi) Because of low leakages in hydraulic actuators, when loads are applied the drop in speed will be small.

(vii) For the same power output, hydraulic motor is much smaller in physical size than an electric motor.

(viii) Hydraulic components are rapidly acting and more rugged compared to the corresponding electrical component.

DISADVANTAGES OF HYDRAULIC DEVICES

(i) Hydraulic power is not readily available compared to electric power.

(ii) They have the inherent problems of leaks and of sealing them against foreign particles.

(iii) Operating noise.

(iv) Costs more when compared to electrical system.

(v) Tendency to become sluggish at low temperature because of increasing viscosity of fluid.

(vi) Fire and explosion hazards exist.

(vii) Hydraulic lines are not flexible as electric cables.

(viii) Because of the non-linear and other complex characteristics involved, it is difficult to design sophisticated hydraulic systems.

EXAMPLE OF HYDRAULIC DEVICE

The most frequently used hydraulic device in control system is hydraulic motor-pump set. It consists of a variable stroke hydraulic pump and a fixed stroke hydraulic motor as shown in fig 1.30. The device accepts a linear displacement (stroke length) as input and delivers a large output torque.

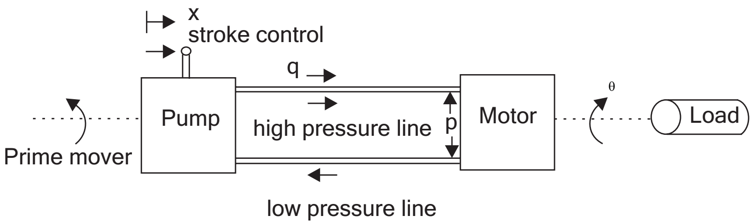


Fig 1.30 : *Hydraulic motor-pump set.*

The hydraulic motor is controlled by the amount of oil delivered by the pump. By mechanically changing the pump stroke, the oil delivered by the pump is controlled. Like in a DC generator and motor, there is no essential difference between hydraulic pump and motor. In a pump the input is mechanical power and output is hydraulic power and in a motor, it is viceversa.

Let, q_p = Rate at which the oil flows from the pump
q_m = Oil flow rate through the motor
q_i = Leakage flow rate
q_c = Compressibility flow rate
x = Input stroke length
θ = Output angular displacement of motor
P = Pressure drop across motor

The rate at which the oil flow from the pump is proportional to stroke angle, i.e., $q_p \propto x$.

$\therefore$ Oil flow rate from the pump, $q_p = K_p\, x$ .....(1.53)

where K_p = Ratio of rate of oil flow to unit stroke angle.

The rate of oil flow through the motor is proportional to motor speed, i.e., $q_m \propto \dfrac{d\theta}{dt}$.

$\therefore$ Oil flow rate through motor, $q_m = K_m \dfrac{d\theta}{dt}$ .....(1.54)

where K_m = Motor displacement constant.

All the oil from the pump does not flow through the motor in the proper channels. Due to back pressure in the motor, a portion of the ideal flow from the pump leaks back past the pistons of motor and pump. The back pressure is the pressure that is built up by the hydraulic flow to overcome the resistance to free movement offered by load on motor shaft.

It is usually assumed that the leakage flow is proportional to motor pressure, i.e. $q_i \propto P$

$\therefore$ Leakage flow rate, $q_i = K_i\, P$ .....(1.55)

where K_i = constant.

The back pressure built up by the motor not only causes leakage flow in the motor and pump but also causes the oil in the lines to compress. Volume compressibility flow is essentially proportional to pressure and therefore the rate of flow is proportional to the rate of change of pressure, i.e. $q_c \alpha \frac{dp}{dt}$

$\therefore$ Compressibility flow rate, $q_c = K_c \frac{dp}{dt}$ .....(1.56)

where K_c = Coefficient of compressibility.

The rate at which the oil flows from the pump is given by sum of oil flow rate through the motor, leakage flow rate and compressibility flow rate.

$$\therefore q_p = q_m + q_i + q_c \qquad .....(1.57)$$

On substituting from equations (1.53) to (1.56) in equation (1.57) we get,

$$K_p\, x = K_m \frac{d\theta}{dt} + K_i\, p + K_c \frac{dp}{dt} \qquad .....(1.58)$$

The torque T_m developed by the motor is proportional to pressure drop and balances load torque.

$\therefore$ Hydraulic motor torque, $T_m = K_t\, P$ .....(1.59)

where K_t is motor torque constant.

If the load is assumed to consist of moment of inertia J and viscous friction with coefficient B, then,

Load torque, $T_l = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt}$ .....(1.60)

Hydraulic power input = $q_m\, P$ .....(1.61)

On substituting for q_m, from equation (1.54) in equation (1.61) we get,

Hydraulic power input = $K_m \frac{d\theta}{dt}\, p$ .....(1.62)

Mechanical power output = $T_m \frac{d\theta}{dt}$ .....(1.63)

On substituting for T_m from equation (1.59) in equation (1.63) we get,

Mechanical power output = $K_t\, p \frac{d\theta}{dt}$ .....(1.64)

If hydraulic motor losses are neglected or included as a part of load, then the hydraulic motor input is equal to mechanical power output of hydraulic motor.

$$\therefore\ K_m \frac{d\theta}{dt}\, p = K_t\, p \frac{d\theta}{dt} \qquad .....(1.65)$$

From equation (1.65), it is clear that, $K_m = K_t$.

Hence, equation (1.59) can be written as

$$T_m = K_t\, P = K_m\, P$$

Since the motor torque equals load torque, $T_m = T_l$

$$\therefore K_m\, P = J\,\frac{d^2\theta}{dt^2} + B\,\frac{d\theta}{dt}$$

$$\therefore \quad P = \frac{J}{K_m}\,\frac{d^2\theta}{dt^2} + \frac{B}{K_m}\,\frac{d\theta}{dt} \qquad .....(1.66)$$

On differentiating the above equation with respect to t, we get,

$$\frac{dp}{dt} = \frac{J}{K_m}\,\frac{d^3\theta}{dt^3} + \frac{B}{K_m}\,\frac{d^2\theta}{dt^2} \qquad .....(1.67)$$

On substituting for P and dP/dt from equations (1.66) and (1.67) in equation (1.58) we get,

Note : $\mathcal{L}\{x\} = X(s)$; $\mathcal{L}\{\theta\} = \theta(s)$; $\mathcal{L}\left\{\frac{d^n\theta}{dt^n}\right\} = s^n\theta(s)$

$$K_p x = K_m\,\frac{d\theta}{dt} + K_1\left[\frac{J}{K_m}\,\frac{d^2\theta}{dt^2} + \frac{B}{K_m}\,\frac{d\theta}{dt}\right] + K_c\left[\frac{J}{K_m}\,\frac{d^3\theta}{dt^3} + \frac{B}{K_m}\,\frac{d^2\theta}{dt^2}\right]$$

$$K_p x = \frac{K_c J}{K_m}\,s^3\,\theta(s) + \left(\frac{K_i J}{K_m} + \frac{K_c B}{K_m}\right)\frac{d^2\theta}{dt^2} + \left(K_m + \frac{K_i B}{K_m}\right)\frac{d\theta}{dt}$$

On taking Laplace transform with zero initial conditions, we get,

$$K_p X(s) = \frac{K_c J}{K_m}\,s^3\theta(s) + \left(\frac{K_i\,J}{K_m} + \frac{K_c B}{K_m}\right)s^2\theta(s) + \left(K_m + \frac{K_i B}{K_m}\right)s\theta(s)$$

$$\therefore \frac{\theta(s)}{X(s)} = \frac{K_p}{s\left[\frac{K_c J}{K_m}\,s^2 + \left(\frac{K_i J + K_c B}{K_m}\right)s + \frac{K_m{}^2 + K_i B}{K_m}\right]} \qquad ..... (1.68)$$

In hydraulic systems, normally $K_c << K_m$, therefore, Put $K_c = 0$, in equation (1.68).

$$\therefore \quad \frac{\theta(s)}{X(s)} = \frac{K_p}{s\left[\frac{K_i J}{K_m}s + \frac{K_m{}^2 + K_i B}{K_m}\right]} = \frac{\dfrac{K_p}{\dfrac{K_m{}^2 + K_i B}{K_m}}}{s\left[\dfrac{K_i J}{K_m{}^2 + K_i B}\,s + 1\right]} = \frac{K}{s(\tau s + 1)} \qquad .....(1.69)$$

$$\text{where,} \quad K = \frac{K_p}{\dfrac{K_m{}^2 + K_i B}{K_m}} \quad \text{and} \quad \tau = \frac{K_i J}{K_m{}^2 + K_i B}$$

The equation (1.69) is the required transfer function of the system.

1.15 PNEUMATIC SYSTEM

Pneumatic system uses compressible fluid as working medium usually air. In pneumatic systems, compressibility effects of gas cannot be neglected and hence dynamic equations are obtained using conservation of mass.

In pneumatic systems, change in fluid inertia energy and the fluid's internal thermal energy are assumed negligible. In pneumatic system, which employs compressible fluid as working fluid, the mass and volume flow rates are not readily interchangeable and the analysis of gas flow is more complicated.

The pneumatic devices involve the flow of gas or air, through connected pipe lines and pressure vessels. Hence the variables of pneumatic system are mass flow rate, q_m and pressure, P. The mass flow rate is through variable and it is analogous to current. The pressure is across variable and it is analogous to voltage.

The two basic elements of pneumatic system are the resistance and capacitance. The restrictions in the pipes and valves offers resistance to gas flow.

The ***gas flow resistance***, R is defined as the rate of change in gas pressure difference for a change in gas flow rate.

$$\boxed{R = \frac{\text{Change in gas pressure difference, N/m}^2}{\text{Change in gas flow rate, Kg/sec}}}$$

The **pnuematic capacitance** is defined for a pressure vessel and depends on the type of expansion process involved. The capacitance of a pressure vessel may be defined as the ratio of change in gas stored for a change in gas pressure.

$$\boxed{C = \frac{\text{Change in gas stored, Kg}}{\text{Change in gas pressure, N/m}^2}}$$

Pneumatic devices are employed in guided missiles,air craft systems, automation of production machinery and in many other fields as automatic controllers.

The advantages of pneumatic system are,

1. The air or gas used is non-inflammable and so it offers safety from fire hazards.
2. The air or gas has negligible viscosity, compared to the high viscosity of hydraulic fluids.
3. No return pipelines are required since air can be let out, at the end of device work cycle.

The disadvantage in pneumatic system is that the response is slower than that of hydraulic systems, because of the compressibility of the working fluid.

EXAMPLE OF A PNEUMATIC SYSTEM

A simple pneumatic system is shown in fig 1.31 and it consist of a ***pneumatic bellows*** in line with the restriction. The pneumatic bellows consists of a hollow chamber with thin pneumatic walls. The side walls of bellows are corrugated and the input and output surface are flat. An increase in pressure within the bellows results in an increase in separation between the input and output surfaces.

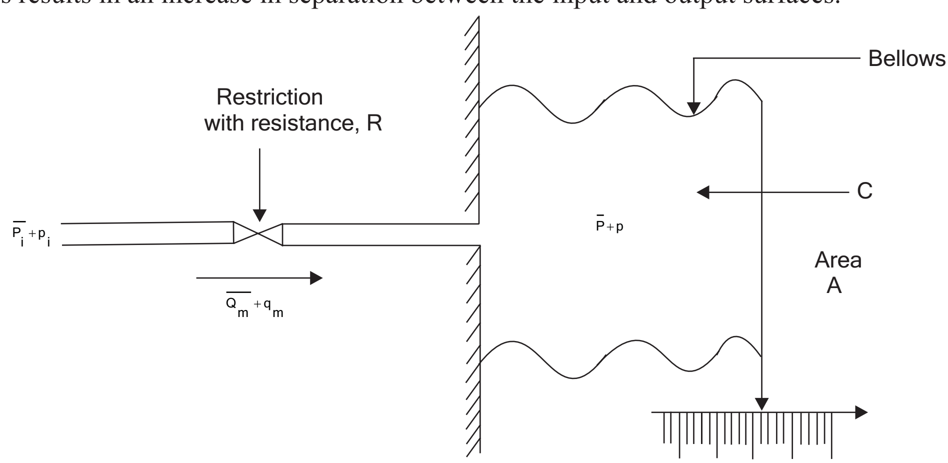


Fig 1.31: *Pneumatic system*

Let, $\overline{P_i}$ = Steady-state value of input air pressure
p_i = Increase in the pressure of air-source
$\overline{P}$ = Steady-state value of pressure inside the bellows
p = Increase in pressure inside the bellow
$\overline{Q_m}$ = Steady-state value of air flow rate
q_m = Increase in air flow rate
A = Area of each flat surface of the bellows
R = Resistance of the restriction
C = Capacitance of the bellows
x = Displacement of the movable surface of the bellows

Let the pressure of air source be increased from its steady state value by an amount p_i. This results in an increase in air flow by q_m and increase in the pressure inside the bellows by p. Due to increase in pressure, there will be a displacement of the movable surface of the bellows, by an amount x. Here, the terms p_i, q_m, p and x are all functions of time, t and so can be expressed as $p_i(t)$, $q_m(t)$, $p(t)$ and $x(t)$.

The force exerted on the movable surface of the bellows is proportional to increase in pressure inside the bellows, i.e, $f_b \alpha p(t)$.

$\therefore$ Force exerted on the movable surface of the bellow, $f_b = A\, p(t)$ .....(1.70)

The force opposing the movement of the flat surface of bellow walls is proportional to displacement i.e, $f_0 \alpha x(t)$.

$\therefore$ Force opposing the motion, $f_0 = K\, x(t)$ .....(1.71)

where K = Constant representing stiffness of bellows.

At steady state the above two forces are balanced,

$$\therefore \quad f_b = f_0$$

$$A\, p(t) = K\, x(t) \qquad .....(1.72)$$

$$\text{The resistance, } R = \frac{\text{Difference between change in pressure}}{\text{Change in air flow rate}} = \frac{p_i(t) - p(t)}{q_m(t)}$$

$$\therefore \quad q_m(t) = \frac{p_i(t) - p(t)}{R} \qquad .....(1.73)$$

$$\text{The capacitance, } C = \frac{\text{Change in air flow rate}}{\text{Rate of change of pressure}} = \frac{q_m(t)}{dp(t)/dt}$$

$$\therefore \quad q_m(t) = C\frac{dp(t)}{dt} \qquad .....(1.74)$$

On equating the two equations of $q_m(t)$ we get,

$$C\frac{dp(t)}{dt} = \frac{p_i(t) - p(t)}{R}$$

$$\therefore \quad RC\frac{dp(t)}{dt} + p(t) = p_i(t) \qquad .....(1.75)$$

From the equation (1.72), we get, $p(t) = \frac{K}{A}x(t)$ .....(1.76)

On differentiating equation (1.76) with respect to t, we get,

$$\frac{dp(t)}{dt} = \frac{K}{A}\frac{dx(t)}{dt} \qquad .....(1.77)$$

On substituting for p(t) and dp(t)/dt from equations (1.76) and (1.77) in equation (1.75) we get,

$$RC\left[\frac{K}{A}\frac{dx(t)}{dt}\right]+\frac{K}{A}x(t)=p_i(t) \qquad .....(1.78)$$

On taking Laplace transform with zero initial conditions, we get,

$$RC\left[\frac{K}{A}s\,X(s)\right]+\frac{K}{A}X(s)=p_i(s)$$

$$\frac{K}{A}(RC\,s+1)X(s)=p_i(s)$$

$$\therefore \quad \frac{X(s)}{p_i(s)}=\frac{A/K}{RC\,s+1}=\frac{A/K}{\tau\,s+1} \qquad .....(1.79)$$

where, τ = RC = Time constant of the system.

The equation (1.79) is the required transfer function of the system.

TABLE - 1.10 : Difference Between Hydraulic and Pneumatic System

Pneumatic	Hydraulic
1. Working fluid is compressible.	1. Working fluid is incompressible.
2. Working fluid lack lubricating property.	2. Working fluid acts as lubricant.
3. Operating pressure is lower.	3. Higher operating pressure.
4. Output power is less.	4. More output power.
5. Accuracy of actuator is poor.	5. More accuracy can be achieved.
6. External leakage is permissible but internal leakage must be avoided.	6. Internal leakage is permissible and external leakage must be avoided.
7. No return pipes are required.	7. Return pipes are required.
8. Insensitive to temperature changes.	8. Sensitive to changes in temperature.
9. Fire and explosion proof.	9. Not a fire and explosion proof.

1.16 SHORT - ANSWER QUESTIONS

Q1.1 What is system?

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system.

Q1.2 What is control system?

A system consists of a number of components connected together to perform a specific function. In a system when the output quantity is controlled by varying the input quantity, then the system is called control system. The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

Q1.3 What are the two major type of control systems?

The two major type of control systems are open loop and closed loop systems.

Q1.4 Define open loop system.

The control system in which the output quantity has no effect upon the input quantity are called open loop control system. This means that the output is not fedback to the input for correction.

Q1.5 Define closed loop system.

The control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called closed loop control systems.

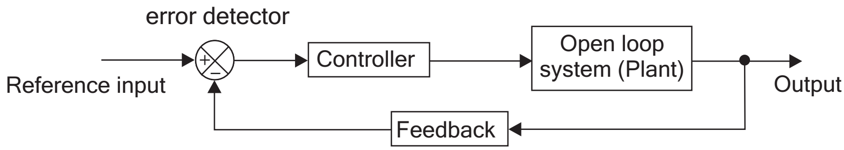
Q1.6 What is feedback? What type of feedback is employed in control system?

The feedback is a control action in which the output is sampled and a proportional signal is given to input for automatic correction of any changes in desired output.

Negative feedback is employed in control system.

Q1.7 What are the components of feedback control system?

The components of feedback control system are plant, feedback path elements, error detector and controller.



Q1.8 Why negative feedback is invariably preferred in a closed loop system?

The negative feedback results in better stability in steady state and rejects any disturbance signals. It also has low sensitivity to parameter variations. Hence negative feedback is preferred in closed loop systems.

Q1.9 What are the characteristics of negative feedback?

The characteristics of negative feedback are as follows :

(i) accuracy in tracking steady state value.

(ii) rejection of disturbance signals.

(iii) low sensitivity to parameter variations.

(iv) reduction in gain at the expense of better stability.

Q1.10. What is the effect of positive feedback on stability?

The positive feedback increases the error signal and drives the output to instability. But sometimes the positive feedback is used in minor loops in control systems to amplify certain internal signals or parameters.

Q1.11. Distinguish between open loop and closed loop system.

Open loop	Closed loop
1. Inaccurate & unreliable.	1. Accurate & reliable.
2. Simple and economical.	2. Complex and costly.
3. Changes in output due to external disturbances are not corrected automatically.	3. Changes in output due to external disturbances are corrected automatically.
4. They are generally stable.	4. Great efforts are needed to design a stable system.

Q1.12 What is servomechanism?

The servomechanism is a feedback control system in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

Q1.13 State the principle of homogenity (or) State the principle of superposition.

The principle of superposition and homogenity states that if the system has responses $c_1(t)$ and $c_2(t)$ for the inputs $r_1(t)$ and $r_2(t)$ respectively then the system response to the linear combination of these input $a_1r_1(t) + a_2r_2(t)$ is given by linear combination of the individual outputs $a_1c_1(t) + a_2c_2(t)$, where a_1 and a_2 are constants.

Q1.14 Define linear system.

A system is said to be linear, if it obeys the principle of superposition and homogenity, which states that the response of a system to a weighed sum of signals is equal to the corresponding weighed sum of the responses of the system to each of the individual input signals. The concept of linear system is diagrammatically shown below.

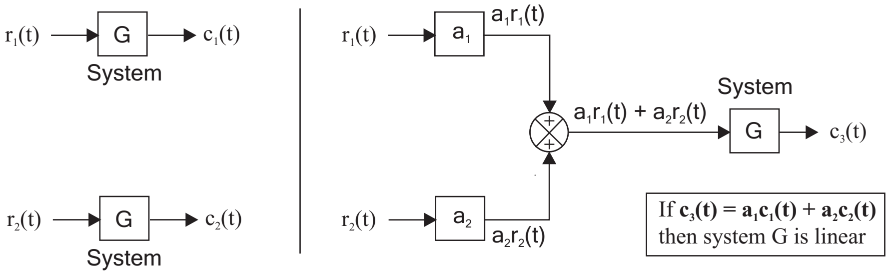


Fig Q1.14 : Principle of linearity and superposition.

Q1.15 What is time invariant system?

A system is said to be time invariant if its input-output characteristics do not change with time. A linear time invariant system can be represented by constant coefficient differential equations. (In linear time varying systems the coefficients of the differential equation governing the system are function of time).

Q1.16 Define transfer function.

The transfer function of a system is defined as the ratio of Laplace transform of output to Laplace transform of input with zero initial conditions. (It is also defined as the Laplace transform of the impulse response of system with zero initial conditions).

Q1.17 State whether transfer function technique is applicable to non-linear system and whether the transfer function is independent of the input of a system.

(i) The transfer function technique is not applicable to non-linear system

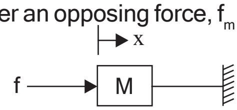
(ii) The transfer function of a system is independent of input and depends only on system parameters but the output of a system depends on input.

Q1.18 What are the basic elements used for modelling mechanical translational system?

The model of mechanical translational system can be obtained by using three basic elements mass, spring and dashpot.

Q1.19 Write the force balance equation of ideal mass element.

Let a force f be applied to an ideal mass M. The mass will offer an opposing force, f_m which is proportional to acceleration.

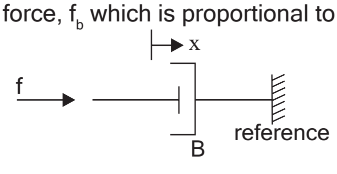


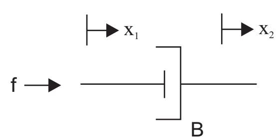
$$\therefore\ f = f_m = M\frac{d^2x}{dt^2}$$

Q1.20 Write the force balance equation of ideal dashpot.

Let a force f be applied to an ideal dashpot, with viscous frictional coefficient B. The dashpot will offer an opposing force, f_b which is proportional to velocity.

$$f = f_b = B\frac{dx}{dt}$$





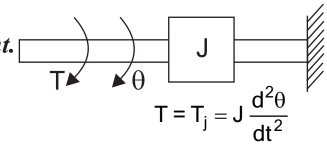
Q1.21 Write the force balance equation of ideal spring.

Let a force f be applied to an ideal spring with spring constant K. The spring will offer an opposing force, f_k which is proportional to displacement.

Q1.22 ***What are the basic elements used for modelling mechanical rotational system?***

The model of mechanical rotational system can be obtained using three basic elements mass with moment of inertia, J, dash-pot with rotational frictional coefficient, B and torsional spring with stiffness, K.

Q1.23 ***Write the torque balance equation of an ideal rotational mass element.***



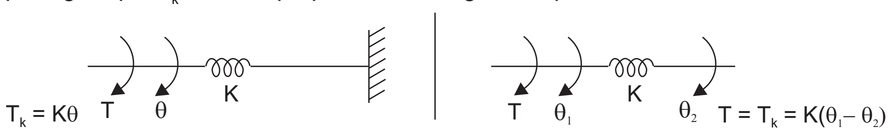
Let a torque T be applied to an ideal mass with moment of inertia, J. The mass will offer an opposing torque T_j which is proportional to angular acceleration.

Q1.24 ***Write the torque balance equation of an ideal rotational dash-pot.***

Let a torque T be applied to a rotational dash-pot with frictional coefficient B. The dashpot will offer an opposing torque which is proportional to angular velocity.

Q1.25 ***Write the torque balance equation of ideal rotational spring.***

Let a torque T be applied to an ideal rotational spring with spring constant K. The spring will offer an opposing torque T_k which is proportional to angular displacement.



Q1.26 ***Name the two types of electrical analogous for mechanical system.***

The two types of analogies for the mechanical system are force-voltage and force-current analogy.

Q1.27 ***Write the analogous electrical elements in force-voltage analogy for the elements of mechanical translational system.***

Force, f	→	Voltage, e	Frictional coefficient , B	→	Resistance, R
Velocity, v	→	Current, i	Stiffness, K	→	Inverse of capacitance, 1/C
Displacement, x	→	Charge, q	Newton's second law, $\Sigma f = 0$	→	Kirchoff 's voltage law, $\Sigma v = 0$
Mass, M	→	Inductance, L			

Q1.28 ***Write the analogous electrical elements in force-current analogy for the elements of mechanical translational system.***

Force, f	→	Current, i	Frictional coefficient, B	→	Conductance, G =1/ R
Velocity, v	→	Voltage, v	Stiffness, K	→	Inverse of Inductance, 1/L
Displacement, x	→	Flux, ϕ	Newton's second law, $\Sigma f = 0$	→	Kirchoff's current law, $\Sigma i = 0$
Mass, M	→	Capacitance, C			

Q1.29 ***Write the analogous electrical elements in torque-voltage analogy for the elements of mechanical rotational system.***

Torque, T	→ Voltage, e	Stiffness of spring, K	→ Inverse of capacitance, 1/C
Angular velocity, ω	→ Current, i	Frictional coefficient, B	→ Resistance, R
Moment of inertia, J	→ Inductance, L	Newton's second law, $\Sigma T = 0$	→ Kirchoff 's voltage law, $\Sigma v = 0$.
Angular displacement, θ	→ Charge, q		

Q1.30 ***Write the analogous electrical elements in torque-current analogy for the elements of mechanical rotational system.***

Torque, T	→ Current, i	Frictional coefficient , B	→ Conductance, G = 1/R
Angular velocity, ω	→ Voltage, v	Stiffness of spring, K	→ Inverse of inductance, 1/L
Angular displacement, θ	→ Flux, ϕ	Newton's second law, $\Sigma T = 0$	→ Kirchoff 's current law, $\Sigma i = 0$
Moment of inertia, J	→ Capacitance, C		

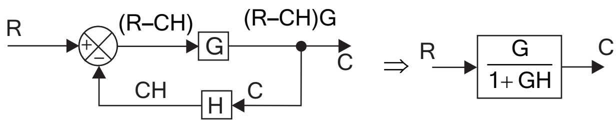
Q1.31 ***What is block diagram? What are the basic components of block diagram?***

A block diagram of a system is a pictorial representation of the functions performed by each component of the system and shows the flow of signals. The basic elements of block diagram are block, branch point and summing point.

Q1.32 ***What is the basis for framing the rules of block diagram reduction technique?***

The rules for block diagram reduction technique are framed such that any modification made on the diagram does not alter the input output relation.

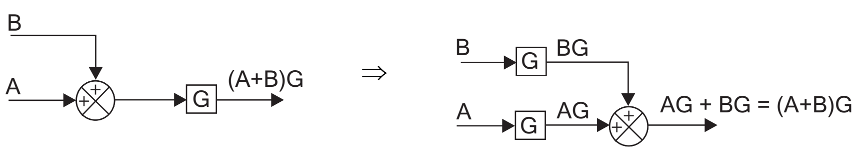
Q1.33 ***Write the rule for eliminating negative feedback loop.***



Proof

$$C = (R - CH)\,G$$
$$C = RG - CHG$$
$$C + CHG = RG$$
$$C(1 + HG) = RG$$
$$\frac{C}{R} = \frac{G}{1+GH}$$

Q1.34 ***Write the rule for moving the summing point ahead of a block.***



Q1.35 ***What is a signal flow graph?***

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

Q1.36 ***What is transmittance?***

The transmittance is the gain acquired by the signal when it travels from one node to another node in signal flow graph.

Q1.37 ***What is sink and source?***

Source is the input node in the signal flow graph and it has only outgoing branches. Sink is a output node in the signal flow graph and it has only incoming branches.

Q1.38 Define non-touching loop.

The loops are said to be non-touching if they do not have common nodes.

Q1.39 What are the basic properties of signal flow graph?

The basic properties of signal flow graph are,

(i) Signal flow graph is applicable to linear systems.

(ii) It consists of nodes and branches. A node is a point representing a variable or signal. A branch indicates functional dependence of one signal on the other.

(iii) A node adds the signals of all incoming branches and transmits this sum to all outgoing branches.

(iv) Signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

(v) The algebraic equations must be in the form of cause and effect relationship.

Q1.40 Write the Mason's gain formula.

Mason's gain formula states that the overall gain of the system [transfer function] as follows,

$$\text{Overall gain, } T = \frac{1}{\Delta}\sum_K P_K \Delta_K$$

T = T(s) = Transfer function of the system

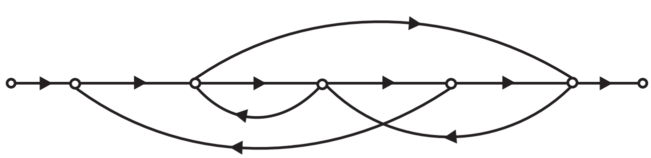
K = Number of forward paths in the signal flow graph

P_K = Forward path gain of K^{th} forward path

$$\Delta = 1 - \begin{bmatrix}\text{sum of individual} \\ \text{loop gains}\end{bmatrix} + \begin{bmatrix}\text{sum of gain products of all possible} \\ \text{combinations of two non-touching loops}\end{bmatrix} - \begin{bmatrix}\text{sum of gain products of all possible} \\ \text{combinations of three non-touching loops}\end{bmatrix} + \ldots\ldots$$

$\Delta_K = \Delta$ for that part of the graph which is not touching K^{th} forward path

Q1.41 For the given signal flow graph, identify the number of forward path and number of individual loop.



Number of forward paths = 2

Number of individual loops = 4

Q1.42 What are the basic elements of thermal system?

The basic elements of thermal system are thermal resistance and thermal capacitance.

Q1.43 Define thermal resistance.

The thermal resistance for heat transfer between two substances is defined as the ratio of change in temparature and change in heat flow rate.

$$\text{Thermal resistance, } R = \frac{\text{Change in temparature,}^\circ\text{C}}{\text{Change in heat flow rate, Kcal/sec}}$$

Q1.44 Define thermal capacitance.

Thermal capacitance is defined as the ratio of change in heat stored and change in temparature.

$$\text{Thermal capacitance, } C = \frac{\text{Change in heat stored, Kcal}}{\text{Change in temperature,}^\circ\text{C}}$$

Q1.45 Mention the electrical analogous of simple thermal system.

The electrical analogous of simple first order thermal system is R-C parallel circuit.

Q1.46 What are the basic elements of hydraulic system?

The basic elements of hydraulic system are resistance, capacitance and inductance.

Q1.47 Define hydraulic resistance.

The resistance for liquid flow is defined as the change in the level difference neccessary to cause the unit change of flow rate.

$$R = \frac{\text{Change in level difference, m}}{\text{Change in flow rate, m}^3/\text{sec}}$$

Q1.48 Define hydraulic capacitance.

The capacitance C of tank is defined to be change in quantity of storedliquid neccessary to cause the unit change in head (height).

$$C = \frac{\text{Change in liquid stored, m}^3}{\text{Change in head, m}}$$

Q1.49 What is inertance?

The inertance represents fluid inertia derived from the inertial forces required to accelerate a fluid in a pipe. It is an energy storing elements. The energy storage due to inertance is negligible compared to capacitance elements.

Q1.50 What are the basic elements of pneumatic systems?

The basic elements of pneumatic systems are pneumatic resistance and pneumatic capacitance.

Q1.51 Define pneumatic resistance.

The gas flow resistance, R is defined as,

$$R = \frac{\text{Change in gas pressure difference, N/m}^2}{\text{Change in gas flow rate, m}^3/\text{sec}}$$

Q1.52 Define pneumatic capacitance.

The capacitance of the pressure vessel is defined as,

$$C = \frac{\text{Change in gas stored, m}^3}{\text{Change in gas pressure, N/m}^2}$$

1.17 EXERCISES

E1.1 For the mechanical system shown in fig E1.1derive the transfer function. Also draw the force-voltage and force-current analogous circuits.

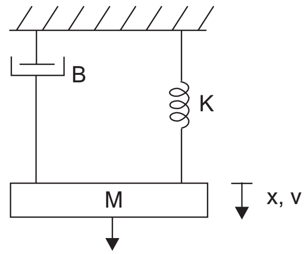


Fig E1.1

E1.2 For the mechanical system shown in fig E1.2 draw the force-voltage and force-current analogous circuits.

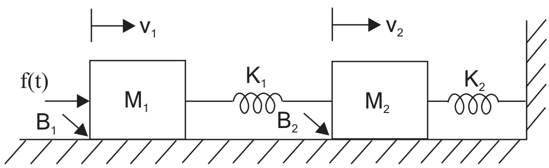


Fig E1.2

E1.3 Write the differential equations governing the mechanical system shown in fig E1.3(a) & (b). Also draw the force-voltage and force-current analogous circuit.

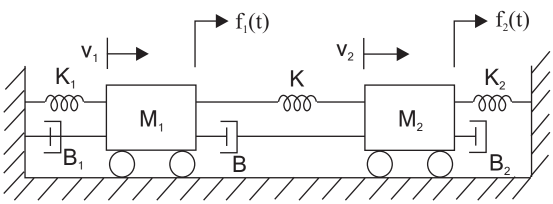


Fig E1.3(a)

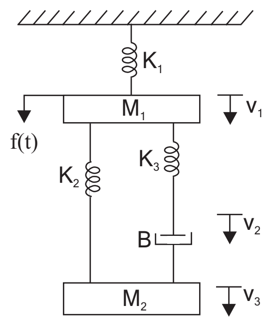


Fig E1.3(b)

E1.4 Consider the mechanical translational system shown in fig E1.4, Draw(a) force-voltage and (b) force-current analogous circuits.

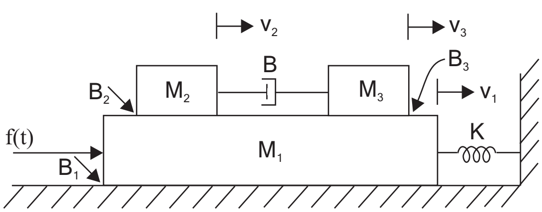


Fig E1.4

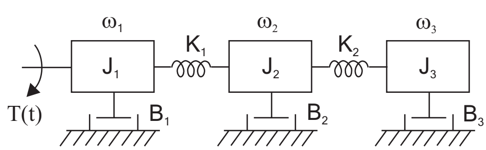


Fig E1.5

E1.5 Write the differential equations governing the rotational mechanical system shown in fig E1.5. Also draw the torque-voltage and torque-current analogous circuits.

E1.6 In an electrical circuit the elements resistance, capacitance and inductance are connected in parallel across the voltage source E as shown in fig E1.6, Draw(a) Translation mechnical analogous system (b) Rotational mechanical analogous system.

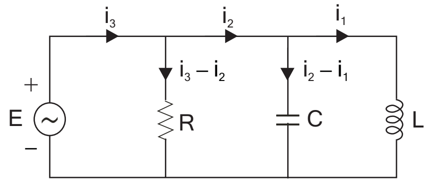


Fig E1.6

E1.7 Consider the block diagram shown in fig E.1.7(a), (b) (c) & (d). Using the block diagram reduction technique, find C/R.

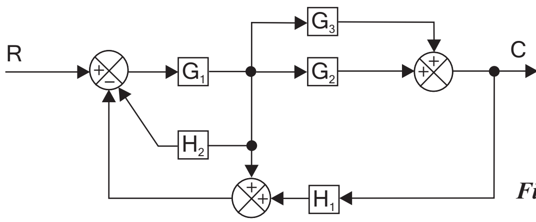


Fig E1.7(a)

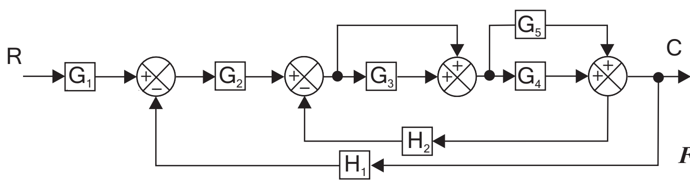


Fig E1.7(b)

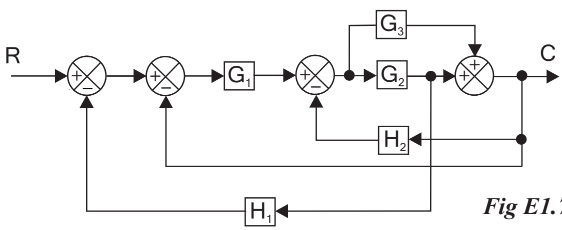


Fig E1.7(c)

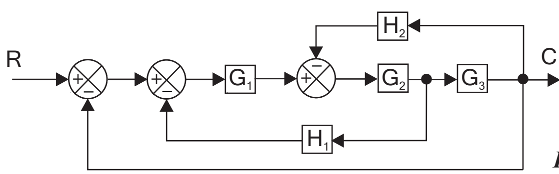


Fig E1.7(d)

E1.8 Convert the block diagram shown in fig E1.8 to signal flow graph and find the transfer function of the system.

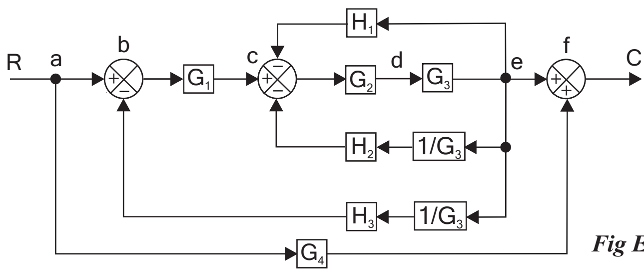


Fig E1.8

E1.9 Consider the system shown in fig E1.9(a), (b), (c) & (d). obtain the transfer function using Mason's gain formula.

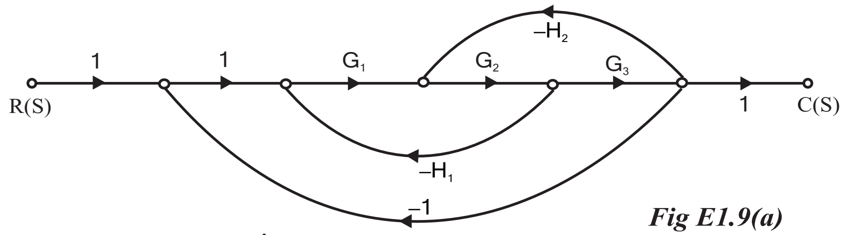


Fig E1.9(a)

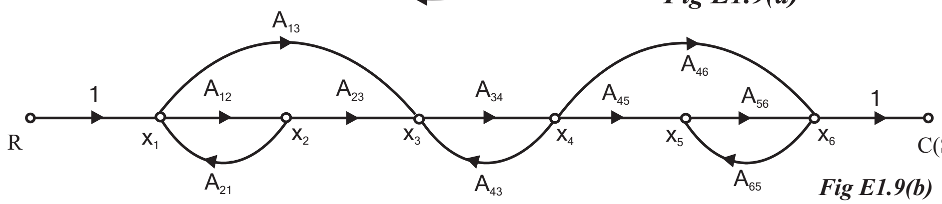


Fig E1.9(b)

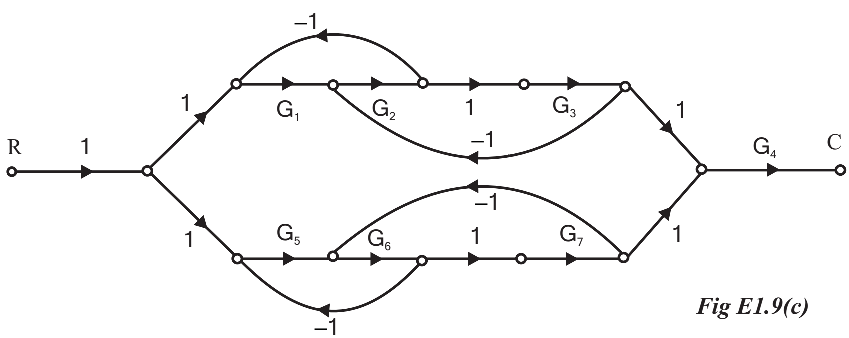


Fig E1.9(c)

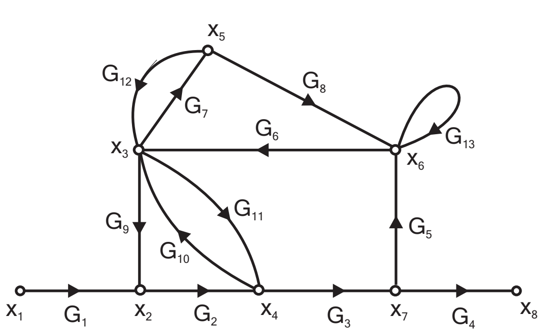


Fig E1.9(d)

E1.10 Consider the signal flow graph shown in fig E.1.10 obtain

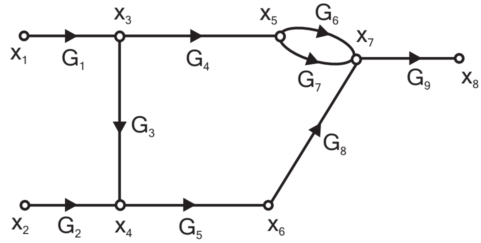


Fig E1.10

E1.11 Find the transfer functions of the networks shown in fig E1.11(a), (b), (c) & (d).

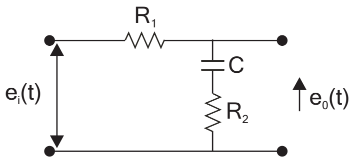


Fig E1.11(a)

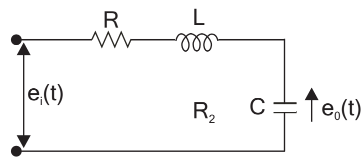


Fig E1.11(b)

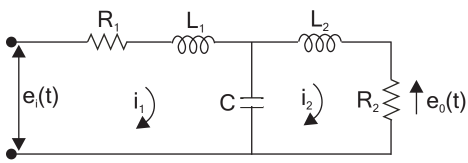


Fig E1.11(c)

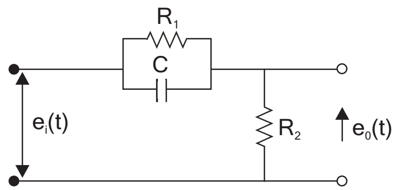


Fig E1.11(d)

E1.12 Find the transfer function of the circuit shown in fig E1.12.

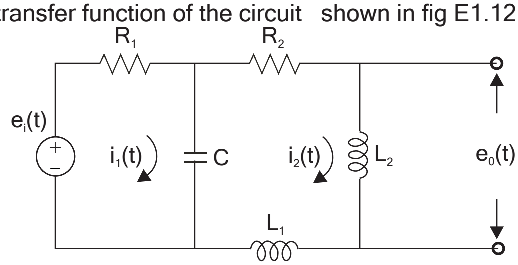


Fig E1.12

E1.13 A process plant consists of two tanks of capacitances C_1 and C_2 respectively. If the flow rate into the top tank is Q_1, find the transfer function relating this flow with liquid level in the bottom tank. Each tank has a resistance R in its outlet pipe. Assume tanks to be non-interacting.

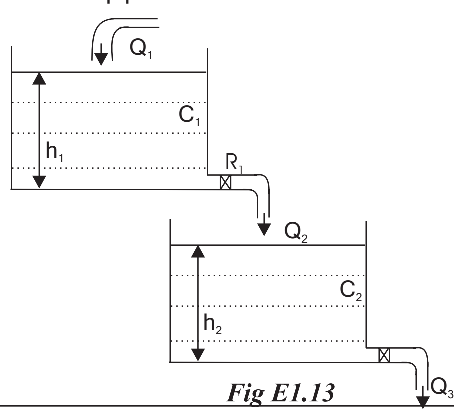
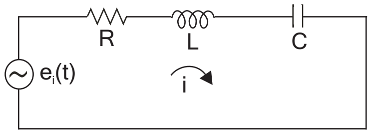


Fig E1.13

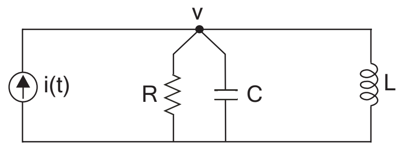
ANSWER FOR EXERCISE PROBLEMS

E1.1 The transfer function is $\dfrac{X(s)}{F(s)} = \dfrac{1}{(Ms^2 + Bs + K)}$



$f(t) \to e(t)$ $M \to L$ $K \to 1/C$
$v \to i$ $B \to R$

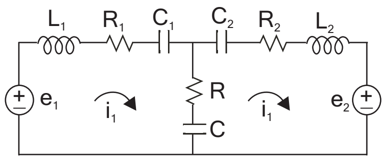
Force-voltage analogous circuit



$f(t) \to i(t)$ $M \to C$ $K \to 1/L$
$v \to v$ $B \to 1/R$

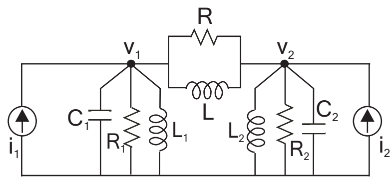
Force-current analogous circuit

E1.2



$f_1 \to e_1$ $M_1 \to L_1$ $B \to R$
$f_2 \to e_2$ $M_2 \to L_2$ $K_1 \to 1/C_1$
$v_1 \to i_1$ $B_1 \to R_1$ $K_2 \to 1/C_2$
$v_2 \to i_2$ $B_2 \to R_2$ $K \to 1/C$

Force-voltage analogous circuit



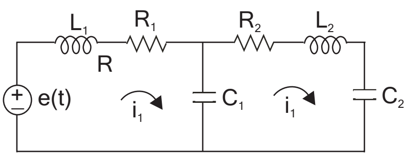
$f_1 \to i_1$ $M_1 \to C_1$ $B \to 1/R$
$f_2 \to i_2$ $M_2 \to C_2$ $K_1 \to 1/L_1$
$v_1 \to v_1$ $B_1 \to 1/R_1$ $K_2 \to 1/L_2$
$v_2 \to v_2$ $B_2 \to 1/R_2$ $K \to 1/L$

Force-current analogous circuit

E1.3(a)

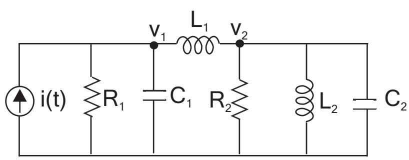
$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B(v_1 - v_2) + K_1 \int v_1 dt + K \int (v_1 - v_2)\, dt = f_1(t)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + B(v_2 - v_1) + K_2 \int v_2 dt + K \int (v_2 - v_1)\, dt = f_2(t)$$



$f(t) \to e(t)$ $M_1 \to L_1$ $B_2 \to R_2$
$v_1 \to i_1$ $M_2 \to L_2$ $K_1 \to 1/C_1$
$v_2 \to i_2$ $B_1 \to R_1$ $K_2 \to 1/C_2$

Force-voltage analogous circuit



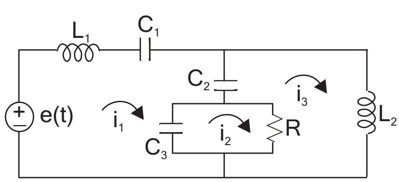
$f(t) \to i(t)$ $M_1 \to C_1$ $B_2 \to 1/R_2$
$v_1 \to v_1$ $M_2 \to C_2$ $K_1 \to 1/L_1$
$v_2 \to v_2$ $B_1 \to 1/R_1$ $K_2 \to 1/L_2$

Force-current analogous circuit

E1.3(b)

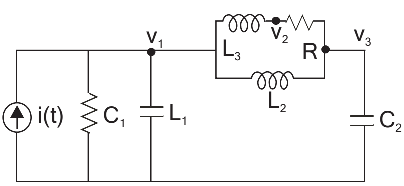
$$M_1 \frac{dv_1}{dt} + .K_1 \int v_1 dt + K_2 \int (v_1 - v_3)\, dt + K_3 \int (v_1 - v_2)\, dt = f(t)$$

$$K_3 \int (v_2 - v_1) dt + B(v_2 - v_3) = 0 \quad ; \quad M_2 \frac{dv_3}{dt} + B(v_3 - v_2) + K_2 \int (v_3 - v_1) dt = 0$$



$f(t) \to e(t)$ $M_1 \to L_1$ $B_1 \to R_1$
$v_1 \to i_1$ $M_2 \to L_2$ $B_2 \to R_2$
$v_2 \to i_2$ $M_3 \to L_3$ $B_3 \to R_3$
$v_3 \to i_3$ $B \to R$ $K \to 1/C$

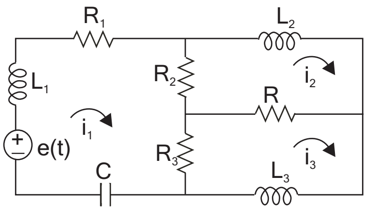
Force-current analogous circuit



$f(t) \to i(t)$ $M_1 \to C_1$ $K_1 \to 1/L_1$
$v_1 \to v_1$ $M_2 \to C_2$ $K_2 \to 1/L_2$
$v_2 \to v_2$ $B \to 1/R$ $K_3 \to 1/L_3$
$v_3 \to v_3$

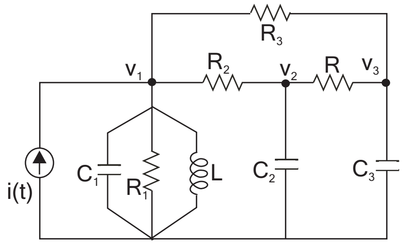
Force voltage-analogous circuit

E1.4



$f(t) \rightarrow e(t)$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$
$v_1 \rightarrow i_1$	$M_2 \rightarrow L_2$	$B_2 \rightarrow R_2$
$v_2 \rightarrow i_2$	$M_3 \rightarrow L_3$	$B_3 \rightarrow R_3$
$v_3 \rightarrow i_3$	$B \rightarrow R$	$K \rightarrow 1/C$

Force voltage-analogous circuit

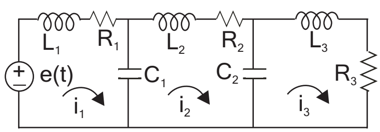


$f(t) \rightarrow i(t)$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$
$v_1 \rightarrow v_1$	$M_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$
$v_2 \rightarrow v_2$	$M_3 \rightarrow C_3$	$B_3 \rightarrow 1/R_3$
$v_3 \rightarrow v_3$	$B \rightarrow 1/R$	$K \rightarrow 1/L$

Force-current analogous circuit

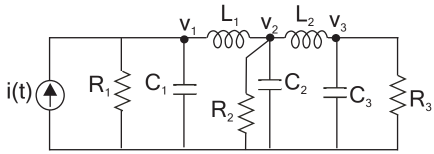
E1.5 $J_1\dfrac{d\omega_1}{dt} + B_1\omega_1 + K_1\int(\omega_1 - \omega_2)dt = T(t);\quad J_2\dfrac{d\omega_2}{dt} + B_2\omega_2 + K_1\int(\omega_2 - \omega_1)dt + K_2\int(\omega_2 - \omega_3)dt = 0$

$$J_3\frac{d\omega_3}{dt} + B_3\omega_3 + K_2\int(\omega_3 - \omega_2)dt = 0$$



$T(t) \rightarrow e(t)$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$\omega_1 \rightarrow i_1$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$\omega_2 \rightarrow i_2$	$J_3 \rightarrow L_3$	$B_3 \rightarrow R_3$	$\omega_3 \rightarrow i$

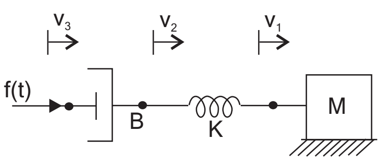
Torque-voltage analogous circuit



$T(t) \rightarrow i(t)$	$J_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$\omega_1 \rightarrow v_1$	$J_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_2 \rightarrow 1/L_2$
$\omega_2 \rightarrow v_2$	$J_3 \rightarrow C_3$	$B_3 \rightarrow 1/R_3$	$\omega_3 \rightarrow v_3$

Torque-current analogous circuit

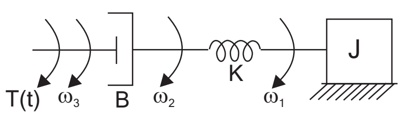
E1.6



$e(t) \rightarrow f(t)$ $\quad i_1 \rightarrow v_1$ $\quad i_3 \rightarrow v_3$ $\quad R \rightarrow B$

$i_2 \rightarrow v_2$ $\quad L \rightarrow M$ $\quad 1/C \rightarrow K$

Analogous mechanical translational system



$e(t) \rightarrow T(t)$ $\quad i_1 \rightarrow \omega_1$ $\quad i_3 \rightarrow \omega_3$ $\quad R \rightarrow B$

$i_2 \rightarrow \omega_2$ $\quad L \rightarrow J$ $\quad 1/C \rightarrow K$

Analogous mechanical rotational system

E1.7 (a) $\dfrac{C}{R} = \dfrac{G_1G_2 + G_1G_3}{1 + G_1H_2 + G_1 + G_1G_2H_1 + G_1G_3H_1}$

(b) $\dfrac{C}{R} = \dfrac{G_1G_2(1+G_3)(G_4+G_5)}{1 + (1+G_3)(G_4+G_5)H_2 + (1+G_3)(G_4+G_5)G_2H_1}$

(c) $\dfrac{C}{R} = \dfrac{G_1G_2 + G_1G_3}{1 + G_1G_2H_1 + G_1G_2 + G_1G_3 + G_2H_2 + G_3H_2}$

(d) $\dfrac{C}{R} = \dfrac{G_1G_2G_3}{1 + G_2G_3H_2 + G_1G_2H_1 + G_1G_2G_3}$

E1.8 $\dfrac{G_1G_2G_3 + G_2G_3G_4H_1 + G_2G_4H_2 + G_1G_2G_4H_3}{1 + G_2G_3H_1 + G_2H_2 + G_1G_2H_3}$

E1.9 (a) $\dfrac{C(s)}{R(s)} = \dfrac{G_1G_2G_3}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$

(b) $$\frac{C}{R} = \frac{A_{12}A_{23}A_{34}A_{45}A_{50} + A_{13}A_{34}A_{45}A_{56} + A_{12}A_{23}A_{34}A_{46} + A_{13}A_{34}A_{46}}{\begin{aligned}1 - (A_{12}A_{21} + A_{34}A_{43} + A_{56}A_{65} + (A_{12}A_{21}A_{34}A_{43} + A_{12}A_{21}A_{56}A_{65})\\ + A_{34}A_{43}A_{56}A_{65}) - (A_{12}A_{21}A_{34}A_{43}A_{56}A_{65})\end{aligned}}$$

(c) $$\frac{C}{R} = \frac{G_1G_2G_3G_4(1 + G_5G_6 + G_6G_7) + G_4G_5G_6G_7(1 + G_1G_2 + G_2G_3)}{1 + [G_1G_2 + G_2G_3 + G_5G_6 + G_6G_7 + G_1G_2G_5G_6 + G_5G_6G_2G_3 + G_1G_2G_6G_7 + G_2G_3G_6G_7}$$

(d) $$\frac{x_8}{x_1} = \frac{[G_1G_2G_3G_4][1 - (G_7G_{12} + G_6G_7G_8 + G_{13}) + G_7G_{12}G_{13}]}{\begin{aligned}1 - [G_2G_9G_{10} + G_{10}G_{11}] + G_2G_3G_5G_6G_9 + G_3G_5G_6G_{11} + G_7G_{12}\\ + G_6G_7G_8 + G_{13}] + G_2G_9G_{10}G_{13} + G_{10}G_{11}G_{13} + G_7G_{12}G_{13}\end{aligned}}$$

E1.10 $\dfrac{x_8}{x_1} = G_1G_4G_6G_9 + G_1G_4G_7G_9 + G_1G_3G_5G_8G_9$; $\dfrac{x_8}{x_1} = G_2G_5G_8G_9$

E1.11 (a) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{1 + sR_2C}{1 + s(R_1 + R_2)C}$

(b) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{1}{s^2LC + sRC + 1}$

(c) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{sR_2C}{(s^2L_1C + sR_1C + 1)(s^2L_2C + sR_2C + 1) - 1}$

(d) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{sR_1R_2C + R_2}{sR_1R_2C + (R_1 + R_2)}$

E1.12 $$\frac{C(s)}{R(s)} = \frac{s^2 L_2 C}{[s R_1 C + 1]\,[s^2 (L_1 + L_2)\, C + s R_2 C + 1] - 1}$$

E1.13 $$\frac{H_2(s)}{Q_1(s)} = \frac{R_2}{(1 + T_1 s)(1 + T_2 s)}$$

where $T_1 = R_1 C_1$ and $T_2 = R_2 C_2$