

∴ Unit vector n perpendicular to vectors a and b is

$$n = \frac{a \times b}{|a \times b|} = \frac{-3i + 5j + 11k}{\sqrt{(-3)^2 + (5)^2 + (11)^2}} = \frac{-3i + 5j + 11k}{\sqrt{155}}$$

Further, we have $|a \times b| = ab \sin \theta$

$$\begin{aligned} \therefore \sin \theta &= \frac{|a \times b|}{ab} = \frac{|-3i + 5j + 11k|}{|2i - j + k| \cdot |3i + 4j - k|} \\ &= \frac{\sqrt{155}}{\sqrt{2^2 + (-1)^2 + (1)^2} \times \sqrt{3^2 + 4^2 + (-1)^2}} \\ &= \sqrt{\frac{155}{\sqrt{6}\sqrt{26}}} = \sqrt{\frac{155}{156}}. \end{aligned}$$

1.6 VECTOR DERIVATIVES

In physics, normally we come across differentiation of a vector with respect to time, i.e. scalar and that with respect to space coordinates x , y and z . Here, we shall discuss both types of derivatives:

(a) Differentiation of a Vector w.r.t Scalar

Let a vector A be a continuous function of a continuous scalar variable t , i.e.

$$A = A(t)$$

If the variable t is increased by Δt , the vector will change by an amount

$$\Delta A = A(t + \Delta t) - A(t)$$

and in complete analogy with scalar functions, we define the derivative $\frac{dA}{dt}$ as the limit

$$A' = \frac{dA}{dt} = \lim_{\Delta t \rightarrow 0} \frac{A(t + \Delta t) - A(t)}{\Delta t} \quad (1.39)$$

Since, division by a scalar does not alter the vectorial properties, the derivative of a vector with respect to a scalar variable is itself a vector.

Derivatives of a higher order are defined similarly, e.g.

$$A'' = \frac{d\left(\frac{dA}{dt}\right)}{dt} = \frac{d^2 A}{dt^2} = \lim_{\Delta t \rightarrow 0} \frac{A'(t + \Delta t) - A'(t)}{\Delta t} \quad (1.40)$$

As an example of differentiation of a vector w.r.t. a scalar, let us consider vector A to be a position vector r joining the origin O of a coordinate system at any point (x, y, z) , then

$$r(t) = x(t)i + y(t)j + z(t)k$$

and specification of the vector function $r(t)$ defines x , y and z functions of t . As t changes, the terminal point of r describes a space curve having parametric equations.

$$x = x(t) \quad y = y(t) \quad z = z(t)$$

1.9 THE CONCEPT OF DIVERGENCE AND GAUSS'S THEOREM

Mathematically, the divergence of a vector A is defined as the scalar product of the vector operator ∇ and the vector A , i.e.

$$\begin{aligned}\operatorname{div} A &= \nabla \cdot A = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (iA_x + jA_y + kA_z) \\ &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\end{aligned}\quad (1.57)$$

We see that $\operatorname{div} A$ is not a vector but is a scalar.

Physical Interpretation

Let us consider flow of a fluid of density $\rho(x, y, z, t)$ and velocity $v = v(x, y, z, t)$ and let A represent the mass of fluid flowing through a unit area normal to the direction of v per unit time. The Y -component of A through the area $ABOC$ (indicated in Fig. 1.18) of infinitesimal element of volume with sides dx , dy , and dz parallel to axes x , y , z in vector field, per unit time is given by $A_y dx dz$.

The flow through the area $DEFG$ per unit time may be represented by the following Taylor's series:

$$A_y(y + dy) dx dz = \left\{ A_y(y) + \frac{\partial A_y}{\partial y} dy + \dots \right\} dx dz$$

We will neglect higher-order terms in the above expansion. The net increase in the mass of the fluid inside the volume element per unit time due to the flow through the two faces is

$$A_y dx dz - \left(A_y + \frac{\partial A_y}{\partial y} dy \right) dx dz = -\frac{\partial A_y}{\partial y} dx dy dz = -\frac{\partial A_y}{\partial y} d\tau$$

Similarly, the net increase in the mass of fluid per unit time due to the flow through $BEFO$ and $ADGC$ is

$$-\frac{\partial A_x}{\partial x} d\tau$$

and that through $FGCO$ and $EDAB$ is

$$-\frac{\partial A_z}{\partial z} d\tau$$

The total increase in mass of fluid per unit volume per unit time due to the excess of inward flow over the outward flow is

$$\frac{\left(-\frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y} - \frac{\partial A_z}{\partial z} \right) d\tau}{d\tau} = -\nabla \cdot A$$

which is just the rate of increase of the density of the fluid inside the volume element $d\tau$. That is to say

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot A \quad (1.58)$$

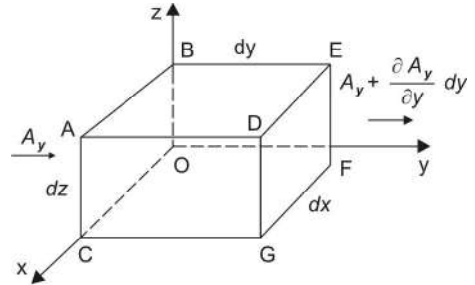


Fig. 1.18

Consider a very small region of such a vector field, several lines of flow of which are shown in (Fig. 1.24), the portion is chosen small enough for the lines to be regarded as nearly straight and parallel. Place a small plane area in this field, shown for convenience as rectangular. If the surface element is placed so that its normal is in the **direction of A** , then A is perpendicular to ds along the entire circuit and the integral vanishes. On the other hand, if the normal is perpendicular to A , then the net contribution of the edges perpendicular to A is zero, but that of the edges parallel to A is not zero since the magnitude of A along the upper edge is assumed to be different from that along the lower edge, the line integral round the boundary has a finite value. The value of the limit

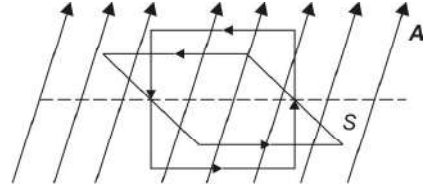


Fig. 1.24

$$L_n = \lim_{\nabla S \rightarrow 1} \oint A \cdot d\mathbf{l} \quad (1.77)$$

is thus a function of the direction of the normal to the surface element.

In general, if we put a small vector area of any shape at any point in a vector field and compute the line integral of the vector A around its bounding edge there will be an orientation of the area for which the line integral is the greatest. The amount of this maximum line integral expressed per unit area is called the curl of the vector field at the point. The curl of a vector field is a vector quantity directed along the normal to the exploring area which is in the position giving the greatest integral.

To calculate the curl in terms of its cartesian components, take three infinitesimal rectangular areas intersecting mutually at right angles at a point where the vector field A has components of magnitudes A_x, A_y, A_z as in Fig. 1.25. Taking the positive normal to the areas along the positive directions of the x -, y - and z -axes respectively, the circular arrows indicate the positive senses in which their boundaries must be traversed to accord with the right-hand screw rule for vector areas.

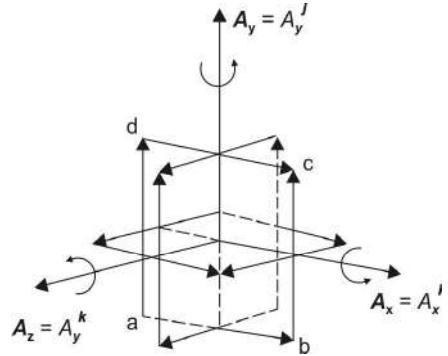


Fig. 1.25

Take one of these areas such as $abcd$ in (Fig. 1.25) with sides dx, dy , its normal being along the axis of z . Since the rectangle is very small, the numerical value of the component of A at the middle of any side may reasonably be taken as the average value for that side; the arrows show the directions in which the components act. Since A_x, A_y, A_z are functions of the coordinates (x, y, z) of the middle of the rectangle, the average value along the four sides ab, bc, dc, ad will be respectively:

$$A_x - \frac{1}{2} \frac{\partial A_x}{\partial y} dy, A_y + \frac{1}{2} \frac{\partial A_y}{\partial x} dx, A_x + \frac{1}{2} \frac{\partial A_x}{\partial y} dy,$$

and

$$A_y - \frac{1}{2} \frac{\partial A_y}{\partial x} dx$$

Integrating, we have

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{c}$$

where $\mathbf{c}$ is a constant vector.

- (b) If $f(r) < 0$, the acceleration has direction opposite to $\mathbf{r}_1$, hence the force is directed towards O and the particle is always attracted towards O.

If $f(r) > 0$, the force is directed away from O and the particle is under the influence of a repulsive force at O.

A force directed towards or away from a fixed point O and having magnitude depending only on the distance r from O is called central force.

- (c) In time ΔT , the particle moves from M to N (Fig. 1.30). The area swept out by the position vector in this time is approximately half the area of parallelogram with sides $\mathbf{r}$ and $\nabla \mathbf{r}$ or $\frac{1}{2} \mathbf{r} \times \Delta \mathbf{r}$. Then the approximate area swept out by the radius vector per unit time is $\frac{1}{2} \mathbf{r} \times \frac{\Delta \mathbf{r}}{\Delta t}$, hence the instantaneous time of rate of change in area is

$$\lim_{\Delta t \rightarrow 0} \frac{1}{2} \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2} \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2} \mathbf{r} \times \mathbf{v}$$

where $\mathbf{v}$ is the instantaneous velocity of the particle. The quantity

$$\mathbf{H} = \frac{1}{2} \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2} \mathbf{r} \times \mathbf{v}$$

is called the areal velocity. From part (a), areal velocity

$$\mathbf{H} = \frac{1}{2} \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \text{constant.}$$

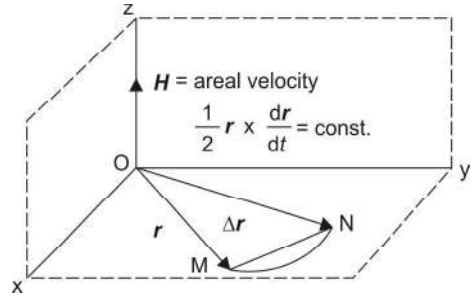


Fig. 1.30

Since $\mathbf{r} \cdot \mathbf{H} = 0$, the motion takes place in plane, which we take as the xy plane as shown in (Fig. 1.30).

- (d) A planet (such as the Earth) is attracted towards the Sun according to Newton's universal law of gravitation, which states that any two objects of mass m and M respectively are attracted towards each other with a force of magnitude $F = \frac{GMm}{r^2}$, where r is the distance between objects and G is a universal constant. Let m and M be the masses of the planet and sun respectively and choose a set of coordinate axis with the origin O at the sun. Then the equation of motion of the planet is

$$m \frac{d^2 \mathbf{r}}{dt^2} = -\frac{GMm}{r^2} \mathbf{r}_1 \quad \text{or} \quad \frac{d^2 \mathbf{r}}{dt^2} = -\frac{GM}{r^2} \mathbf{r}_1$$

assuming the influence of the other objects to be negligible.

According to part (c), a planet moves around the sun, so that its position vector sweeps out equal area in equal time. This result is one of Kepler's famous three laws which he deduced empirically from volumes of data compiled by, Tycho Brahe (an astronomer). These laws enabled Newton to formulate his universal law of gravitation.