8

Periodic structure and filters

Waveguides and transmission lines loaded at periodic intervals with identical obstacles, e.g., a reactive element such as a diaphragm, are referred to as periodic structures. The interest in waveguiding structures of this type arises from two basic properties common to all periodic structures, namely (1) passband-stopband characteristics, and (2) support of waves with phase velocities much less than the velocity of light. The passband-stopband characteristic is the existence of frequency bands throughout which a wave propagates unattenuated (except for incidental conductor losses) along the structure separated by frequency bands throughout which the wave is cut off and does not propagate. The former is called a passband, and the latter is referred to as a stopband. The passband-stopband property is of some interest for its frequency filtering aspects.

The ability of many periodic structures to support a wave having a phase velocity much less than that of light is of basic importance for traveling-wave-tube circuits. In a traveling-wave tube, efficient interaction between the electron beam and the electromagnetic field is obtained only if the phase velocity is equal to the beam velocity. Since the latter is often no greater than 10 to 20 percent of the velocity of light, considerable slowing down of the electromagnetic wave is required. Periodic structures suitable for use in traveling-wave tubes are discussed in this chapter. The actual principles of operation of the tube are covered in Chap. 9.

The last part of the chapter is devoted to an introduction to microwave filter theory. A complete treatment of all aspects of filter theory and design would be much too lengthy to include in this text. However, sufficient material is covered to provide a background so that the technical literature can be read without difficulty.

8.1 CAPACITIVELY LOADED TRANSMISSION-LINE-CIRCUIT ANALYSIS

To introduce a number of basic concepts, methods of analysis, and typical properties of periodic structures, we shall consider a simple example of a capacitively loaded transmission line. For a physically smooth transmission line, such as a coaxial line, the phase velocity is given by

$$\upsilon_p = (LC)^{-\frac{1}{2}} = \left(\mu_0 \kappa \in_0\right)^{-\frac{1}{3}} \tag{8.1}$$

where κ is the dielectric constant of the medium surrounding the conductor. A significant reduction in phase velocity can be achieved in a smooth line only by increasing κ. This method has the great disadvantage that the cross-sectional dimensions of the line must also be reduced to avoid the propagation of higher-order modes. The phase velocity cannot be decreased by increasing the shunt capacity C per unit length because any change in the line configuration to increase C automatically decreases the series inductance L per unit length, since $LC = \mu_0 \in$. However, by removing the restriction that the line should be physically smooth, an effective increase in the shunt capacitance per unit length can be achieved without a corresponding decrease in the series inductance L. That is, lumped shunt capacitance may be added at periodic intervals without affecting the value of L. If the spacing between the added lumped capacitors is small compared with the wavelength, it may be anticipated that the line will appear to be electrically smooth, with a phase velocity

$$\upsilon_p = \left[\left(C + \frac{C_0}{d}\right)L\right]^{-\frac{1}{2}} \tag{8.2}$$

where C_0/d is the amount of lumped capacitance added per unit length (a capacitor C_0 added at intervals d). The following analysis will verify

One method of obtaining shunt capacitive loading of a coaxial transmission line is to introduce thin circular diaphragms at regular intervals, as in Fig. 8.1. The diaphragms may be machined as an integral part of the center conductor. The fringing electric field in the vicinity of the

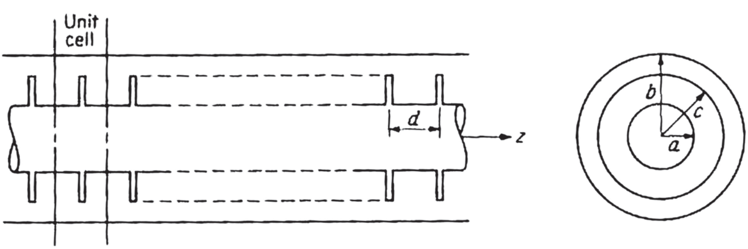

Fig. 8.1: Capacitive loading of a coaxial line by means of thin circular diaphragms.

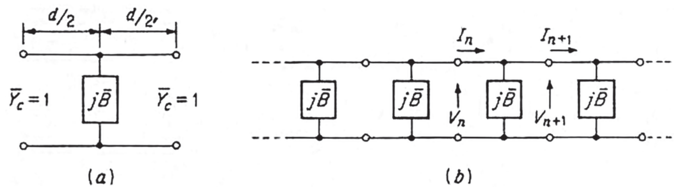

Fig. 8.2: (*a*) Equivalent circuit for unit cell of loaded coaxial line; (*b*) cascade connection of basic unit-cell networks.

diaphragm increases the local storage of electric energy and hence may be accounted for, from a circuit viewpoint, by a shunt capacitance. The local field can be described in terms of the incident, reflected, and transmitted dominant TEM mode and a superposition of an infinite number of higher-order E modes. If the cylinder spacing $b - a$ is small compared with the wavelength, the higher-order modes are evanescent and decay to a negligible value in a distance of the order of $b - a$ away from the diaphragm in either direction. An approximate expression for the shunt susceptance of the diaphragm is †

$$\bar{B} = \frac{B}{Y_c} = \frac{8(b-c)^2}{\lambda_0 c} \frac{\ln(b/a)}{[\ln(b/c)]^2} \ln \csc\left(\frac{\pi}{2}\frac{b-c}{b-a}\right) \tag{8.63}$$

where $Y_c = [60 \ln (b/a)]^{-1}$ is the characteristic admittance of an air-filled coaxial line. The expression for $\bar{B}$ is accurate for $b - a \leq 0.1\lambda_0$. In this low-frequency region, $\bar{B}$ has a frequency dependence directly proportional to ω. At higher frequencies $\bar{B}$ will have a more complicated frequency dependence, although the thin diaphragm can still be represented by a shunt susceptance.

The circuit, or network, analysis of a periodic structure involves constructing an equivalent network for a single basic section or unit cell of the structure first. This is followed by an analysis to determine the voltage and current waves that may propagate along the network consisting of the cascade connection of an infinite number of the basic networks. For the structure of Fig. 8.1 an equivalent network of a basic section is a shunt normalized susceptance $\bar{B}$ with a length $d/2$ of transmission line on either side, as in Fig. 8.2*a*. Figure 8.2*b* illustrates the voltage-current relationships at the input and output of the nth section in the infinitely long cascade connection.

The relationships between the input variables V_n, I_n and the output variables V_{n+1}, I_{n+1} are readily found by using the *abcd* transmission matrix discussed in Sec. 4.9. The V_n and I_n are the total voltage and current amplitudes, i.e., the sum of the contributions from the incident and reflected TEM waves at the terminal plane. The circuit for a unit cell may be broken

† N. Marcuvitz (ed.), "Waveguide Handbook," p. 229, McGraw-Hill Book Company, New York, 1951.

down into three circuits in cascade, namely, a section of transmission line of length $d/2$ (electrical length $\theta/2 = k_0 d/2$), followed by a shunt susceptance $\overline{B}$, which in turn is followed by another length of transmission line. The *abcd* matrix for each of these individual networks is, respectively (Prob. 4.18),

$$\begin{bmatrix} \cos\frac{\theta}{2} & j\sin\frac{\theta}{2} \\ j\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ j\overline{B} & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & j\sin\frac{\theta}{2} \\ j\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

The transmission matrix for the unit cell is obtained by the chain rule [see (4.75)], i.e., the product of the above three matrices, and hence we have

$$\begin{bmatrix} V_n \\ I_n \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} & j\sin\frac{\theta}{2} \\ j\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ j\overline{B} & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & j\sin\frac{\theta}{2} \\ j\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} V_{n+1} \\ I_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta - \frac{\overline{B}}{2}\sin\theta & j\left(\frac{\overline{B}}{2}\cos\theta + \sin\theta - \frac{\overline{B}}{2}\right) \\ j\left(\frac{\overline{B}}{2}\cos\theta + \sin\theta + \frac{\overline{B}}{2}\right) & \cos\theta - \frac{\overline{B}}{2}\sin\theta \end{bmatrix} \begin{bmatrix} V_{n+1} \\ I_{n+1} \end{bmatrix}$$

Note that $a = d$, which is always true for a symmetrical network, i.e., a symmetrical unit cell.

If the periodic structure is capable of supporting a propagating wave, it is necessary for the voltage and current at the $(n + 1)$ st terminal to be equal to the voltage and current at the nth terminal, apart from a phase delay due to a finite propagation time. Thus we assume that

$$V_{n+1} = e^{-\gamma d} V_n \tag{8.5a}$$

$$I_{n+1} = e^{-\gamma d} I_n \tag{8.5b}$$

where $\gamma = j\beta + \alpha$ is the propagation constant for the periodic structure. In terms of the transmission matrix for a unit cell, we now have

$$\begin{bmatrix} V_n \\ I_n \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} V_{n+1} \\ I_{n+1} \end{bmatrix} = e^{\gamma d} \begin{bmatrix} V_{n+1} \\ I_{n+1} \end{bmatrix}$$

or

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e^{\gamma d} & 0 \\ 0 & e^{\gamma d} \end{bmatrix} \right) \begin{bmatrix} V_{n+1} \\ I_{n+1} \end{bmatrix} = 0 \tag{8.6}$$

This equation is a matrix eigenvalue equation for γ. A nontrivial solution for

V_{n+1}, I_{n+1} exists only if the determinant vanishes. Hence

$$\begin{vmatrix} a - e^{\gamma d} & b \\ c & d - e^{\gamma d} \end{vmatrix} = ad - bc + e^{2\gamma d} - e^{\gamma d}(a + d) = 0 \tag{8.7}$$

For a reciprocal network the determinant $ad - bc$ of the transmission matrix equals unity (Sec. 4.9); so we obtain

$$\cosh \gamma d = \frac{a + d}{2} \tag{8.8}$$

For the capacitively loaded coaxial line, (8.8), together with (8.4), yields

$$\cosh \gamma d = \cos\theta - \frac{\overline{B}}{2}\sin\theta \tag{8.9}$$

When $|\cos\theta - (\overline{B}/2)\sin\theta| < 1$, we must have $\gamma = j\beta$ and $\alpha = 0$; that is,

$$\cos \beta d = \cos\theta - \frac{\overline{B}}{2}\sin\theta$$

When the right-hand side of (8.9) is greater than unity, $\gamma = \alpha$ and $\beta = 0$; so

$$\cosh \alpha d = \cos\theta - \frac{\overline{B}}{2}\sin\theta > 1$$

Finally, when the right-hand side of (8.9) is less than –1, we must have $\gamma d = j\pi + \alpha$, so that

$$\cos \gamma d = \cosh (j\pi + \alpha d) = -\cosh \alpha d$$

$$= \cos\theta - \frac{\overline{B}}{2}\sin\theta < -1 \tag{8.10c}$$

It is apparent, then, that there will be frequency bands for which unattenuated propagation can take place separated by frequency bands in which the wave is attenuated. Note that propagation in both directions is possible since $-\gamma$ is also a solution.

A detailed study of the passband-stopband characteristic is made in Sec. 8.6. For the present we shall confine our attention to the low-frequency limiting value of β. When $d \ll \lambda_0$, $\theta = k_0 d$ is small, and βd will then also be small. Replacing $\cos\theta$ by $1 - \theta^2/2$ and $\sin\theta$ by θ in (8.10a) gives

$$\cos\beta\, d \approx 1 - \frac{\beta^2 d^2}{2} = 1 - \frac{k_0^2 d^2}{2} - \frac{\overline{B} k_0 d}{2}$$

Using the relations $k_0^2 = \omega^2 \mu_0 \epsilon_0 = \omega^2 LC$ and $\overline{B} = B/Y_c = \omega C_0 (L/C)^{\frac{1}{2}}$, where $\omega C_0 = B$, we obtain

$$\beta^2 = \omega^2 LC + \frac{\omega^2 L C_0}{d}$$

and hence

$$\beta = \omega\sqrt{L\left(C + \frac{C_0}{d}\right)} \tag{8.11}$$

Therefore we find that, at low frequencies where $d \ll \lambda_0$, the loaded line behaves as an electrically smooth line with a shunt capacitance $C + C_0/d$ per unit length. The increase in β results in a reduction of the phase velocity by a factor k_0/β.

Another parameter of importance in connection with periodic structures is the normalized characteristic impedance $\overline{Z}_B$ presented to the voltage and current waves at the reference terminal plane, i.e., input terminals of a unit cell. An expression for $\overline{Z}_B$ may be obtained from (8.6), which may be written as

$$(a - e^{\gamma d})V_{n+1} = -bI_{n+1}$$
$$-cV_{n+1} = (d - e^{\gamma d})I_{n+1}$$

Hence

$$\frac{Z_B}{ü_c} = \overline{Z}_B = \frac{V_{n+1}}{\quad_{n+1}} = \frac{-ü}{a - e^{\gamma d}} = -\frac{- \quad {}^{\gamma d}}{\qquad} \tag{8.12}$$

Replacing $2e^{\gamma d}$ by $a + d \pm [(a+d)^2 - 4]^{\frac{1}{2}}$ from (8.7), we obtain

$$\overline{Z}_{B^{\pm}} = \frac{2b}{d - a \pm \sqrt{ü a + d \;{}^2 -}} \tag{8.13a}$$

where the upper and lower signs refer to propagation in the $+z$ and $-z$ directions, respectively. We are using the convention that the positive directions of V_n and I_n are those indicated in Fig. 8.2, independent of the direction of propagation. For a symmetrical network, $a = d$, and since $ad - bc = 1$, we have $a^2 - 1 = BC$. In this case (8.13a) reduces to

$$\overline{Z}_{B^{\pm}} = \frac{2b}{\pm\sqrt{4a^2 - 4}} = \pm\sqrt{\frac{b}{c}} \tag{8.13b}$$

In general, for a lossless structure, $\overline{Z}_{B^-} = -(\overline{Z}_{B^+})^*$ in the passband, since $|a + d| < 2$, as (8.8) shows.

If the unit cell is represented by a T network with parameters $\overline{Z}_{11}$, $\overline{Z}_{12}$, and $\overline{Z}_{22}$, then, by using the relations between the *abcd* parameters and the impedance parameters given in Sec. 4.9, we can also show that

$$\cosh \gamma d = \frac{\overline{Z}_{11} + \overline{Z}_{22}}{2\overline{Z}_{12}} \tag{8.14}$$

$$\overline{Z}_B = \frac{\overline{Z}_{11} - \overline{Z}_{22}}{2} \pm \overline{Z}_{12} \sinh \gamma d \tag{8.15}$$

The waves that may propagate along a periodic structure are often called Bloch waves by analogy with the quantum-mechanical electron waves that may propagate through a periodic crystal lattice in a solid. It is for this reason that we have denoted the characteristic impedance as $\bar{Z}_B$ for the Bloch wave. The voltage and current at the nth terminal plane will be denoted by $V_{Bn}^{\pm}$, $I_{Bn}^{\pm}$ for the Bloch waves from now on instead of by the quantities V_n, I_n. The + and – signs refer to Bloch waves propagating in the $+z$ and $-z$ directions. We shall also adopt the convention that the positive direction of current flow for Bloch waves is always in the $+z$ direction; thus $I_{B^+} = \bar{Y}_{B^+}V_{B^+}$ and $I_{B^-} = \bar{Y}_{B^-}V_{B^-}$. However, for a symmetrical structure such that $a = d$, we shall have $\bar{Y}_{B^-} = -\bar{Y}_{B^+} = -(\bar{Z}_{B^+})^{-1}$.

If (8.13) is used, we find that, for the loaded coaxial line,

$$\bar{Z}_B = \sqrt{\frac{b}{c}} = \sqrt{\frac{2\sin\theta + \bar{B}\cos\theta - \bar{B}}{2\sin\theta + \bar{B}\cos\theta + \bar{B}}} \tag{8.16}$$

In the low-frequency limit, where we can replace sin θ by

$$\theta = k_0 d = \omega d\sqrt{LC}$$

and $\cos\theta$ by 1 , we obtain

$$\bar{Z}_B = \sqrt{\frac{2\theta}{2\theta + 2\bar{B}}} = \sqrt{\frac{C}{C + C_0/d}}$$

and thus

$$Z_B = \bar{Z}_B Z_c = \sqrt{\frac{L}{C + C_0/d}} \tag{8.17}$$

Again we see that, in the low-frequency limit, the loaded line is electrically smooth and the characteristic impedance is modified in the anticipated manner by the effective increase in the shunt capacitance per unit length.

The characteristic impedance of a periodic structure is not a unique quantity since it depends on the choice of terminal planes for a unit cell. If the terminal planes are shifted a distance l in the $-z$ direction, the new characteristic impedance becomes

$$\bar{Z}_B' = \frac{\bar{Z}_B + j\tan k_0 l}{1 + j\bar{Z}_B \tan k_0 l} \tag{8.18}$$

8.2 WAVE ANALYSIS OF PERIODIC STRUCTURES

Periodic structures may be analyzed in terms of the forward- and back-ward-propagating waves that can exist in each unit cell with about the same facility

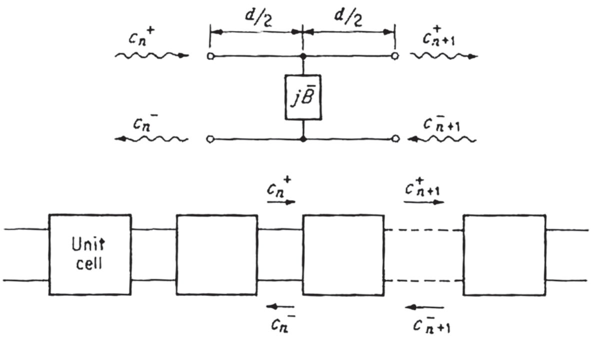

Fig. 8.8: Wave amplitudes in a periodic structure.

as the network approach gives. In the wave approach the wave-amplitude transmission matrix [A] discussed in Sec. 4.9 is used.

With reference to Fig. 8.3, let the amplitudes of the forward- and backward-propagating waves at the n th and (n + 1)st terminal plane be c_n^+, c_n^-, c_{n+1}^+, and c_{n+1}^-. The c_{n+1}^+, c_{n+1}^- are related to the c_n^+, c_n^- by the wave-amplitude transmission matrix as follows:

$$\begin{bmatrix} c_n^+ \\ c_n^- \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} c_{n+1}^+ \\ c_{n+1}^- \end{bmatrix} \tag{8.19}$$

The solution for a Bloch wave requires $c_{n+1}^+ = c_n^+$ and $c_{n+1}^- = e^{-\gamma d} c_n^-$. Hence (8.19) becomes

$$\begin{bmatrix} \ddot{u}_{11} - \quad {}^{\gamma d} & 12 \\ \ddot{u}_{21} & 22 - \quad {}^{\gamma d} \end{bmatrix} \begin{bmatrix} c_{n+1}^+ \\ c_{n+1}^- \end{bmatrix} = 0 \tag{8.20}$$

A nontrivial solution for c_{n+1}^+, c_{n+1}^- is obtained only if the determinant vanishes. Consequently, the eigenvalue equation for γ is

$$A_{11}A_{22} - A_{12}A_{21} + e^{2\gamma d} - e^{\gamma d}(A_{11} + A_{22}) = 0$$

or

$$\cosh \gamma d = \frac{A_{11} + A_{22}}{2} \tag{8.21}$$

since the determinant of the transmission matrix, that is, $A_{11}A_{22} - A_{12}A_{21}$, equals 1 when normalized wave amplitudes are used.

The Bloch wave which can propagate in the periodic structure is made up from forward- and backward-propagating normal transmission-line or waveguide waves that exist between discontinuities. When γ has been determined from (8.21), the ratio of c_n^- to c_n^+ is fixed. This ratio is called the characteristic reflection coefficient Γ_B. Thus the transverse electric field of the Bloch wave will have an amplitude

$$V_{B_0} = c_{\ddot{u}}^+ + c^- = c^+ \ddot{u} + \Gamma_B$$

at the zeroth terminal plane and an amplitude

$$V_{B_n} = c_n^+ + c_n^- = c_n^+(1+\Gamma_B) = c_0^+(1+\Gamma_B)e^{-\gamma nd} \tag{8.22a}$$

at the nth terminal plane. The transverse magnetic field of the Bloch wave will have an amplitude

$$I_{B_n} = c_0^+(1-\Gamma_B)e^{-\gamma nd} \tag{8.22b}$$

at the nth terminal plane.

The characteristic reflection coefficient may be found from the pair of equations (8.20) by eliminating $e^{-\gamma d}$ by the use of (8.21). It is usually more convenient to express Γ_B in terms of $\overline{Z}_B$ by using the relation $\overline{Z}_B = (1 + \Gamma_B)/(1 - \Gamma_B)$. Thus we have

$$\Gamma_{B^\pm} = \frac{\overline{Z}_{B^\pm} - 1}{\overline{Z}_{B^\pm} + 1} \tag{8.23}$$

where the + and – signs refer to Bloch waves propagating in the $+z$ and $-z$ directions, respectively.

The above wave formulation is now applied to the capacitively loaded transmission line discussed earlier. The unit cell is chosen as in Fig. 8.3. The wave-amplitude transmission matrices for the three sections of the unit cell are (Sec. 4.9 and Prob. 8.7)

$$\begin{bmatrix} e^{jk_0 d/2} & 0 \\ 0 & e^{-jk_0 d/2} \end{bmatrix} \begin{bmatrix} \dfrac{2+j\overline{B}}{2} & j\dfrac{\overline{B}}{2} \\ -j\dfrac{\overline{B}}{2} & \dfrac{4+\overline{B}^2}{2\left(2+j\overline{B}\right)} \end{bmatrix}$$

and another matrix like the first one. The $[A]$ matrix for the unit cell is obtained by multiplying the three component matrices together; thus

$$[A] = \begin{bmatrix} e^{j\theta/2} & 0 \\ 0 & e^{-j\theta/2} \end{bmatrix} \begin{bmatrix} \dfrac{2+j\overline{B}}{2} & j\dfrac{\overline{B}}{2} \\ -j\dfrac{\overline{B}}{2} & \dfrac{4+\overline{B}}{2\left(2+j\overline{B}\right)} \end{bmatrix} \begin{bmatrix} e^{j\theta/2} & 0 \\ 0 & e^{-j\theta/2} \end{bmatrix}$$

where $\theta = k_0 d$. After multiplication we obtain

$$[A] = \begin{bmatrix} \dfrac{2+j\overline{B}}{2} e^{j\theta} & j\dfrac{\overline{B}}{2} \\ -j\dfrac{\overline{B}}{2} & \dfrac{4+\overline{B}^2}{2\left(2+j\overline{B}\right)} e^{-j\theta} \end{bmatrix} \tag{8.24}$$

Making use of (8.21), we find that

$$\cosh \gamma d = \frac{(4+\overline{B}^2)e^{-j\theta} + (2+j\overline{B})^2 e^{j\theta}}{4(2+j\overline{B})} = \cos\theta - \frac{\overline{B}}{2}\sin\theta$$

which is the same as (8.9) obtained earlier.